AN AHLFORS ISLANDS THEOREM FOR NON-ARCHIMEDEAN MEROMORPHIC FUNCTIONS

ROBERT L. BENEDETTO

Abstract. We present a p-adic and non-archimdean version of the Five Islands Theorem for meromorphic functions from Ahlfors' theory of covering surfaces. In the non-archimedean setting, the theorem requires only four islands, with explicit constants. We present examples to show that the constants are sharp and that other hypotheses of the theorem cannot be removed. This paper extends an earlier theorem of the author for holomorphic functions.

In the 1930s, Ahlfors proposed his theory of covering surfaces [2] in complex analysis as an analogue of Nevanlinna theory for domains, rather than for points. The Ahlfors theory allows for a description of the mapping properties of complex meromorphic functions with respect to open subsets of the image. One of the key theorems in the subject is the Five Islands Theorem:

Theorem. (Ahlfors' Complex Five Islands Theorem) *Let $U_1, \ldots, U_5$ be simply connected domains in the Riemann sphere with mutually disjoint closures. Then there is a constant $h = H(U_1, \ldots, U_5) > 0$ with the following property: Let f be a complex meromorphic function on the disk $|z| < 1$, and suppose that there is some $r \in (0, 1)$ with*

$$S(f, r) \geq h \cdot L(f, r). \tag{0.1}$$

Then there is a simply connected domain U contained in the disk $|z| < R$ such that f is one-to-one on U and $f(U) = U_i$ for some $1 \leq i \leq 5$.

Here, $S(f, r)$ and $L(f, r)$ (the *mean covering number* and *relative boundary length* of f, respectively) are certain real quantities describing the image of f on the open disk $|z| < r$. By the work of Dufresnoy [16], condition (0.1) may be replaced by a condition of the form $f^{\#}(0) > \tilde{h}$, where $f^{\#}$ is the spherical derivative of f, and $\tilde{h}$ is, like h, a constant which depends only on the domains $U_1, \ldots, U_5$. Similar results hold for holomorphic functions, with only three islands $U_i \subseteq \mathbb{C}$ required. Recently, Bergweiler [7] proved the Five Islands Theorem without the theory of covering surfaces by using a lemma of Zalcman [29], some Nevanlinna theory, and quasiconformal perturbations. See [19], Chapters 5–6, for more details on the theory of covering surfaces.

Initially, the Five Islands Theorem was used mainly in complex function theory. Then, in 1968, Baker [3] applied it in the study of complex dynamics to prove repelling density for entire functions. That is, he proved that the Julia set of a complex entire function

Date: July 7, 2004.
2000 *Mathematics Subject Classification.* Primary: 30G06 Secondary: 30D35, 11J97, 11Q25.
Key words and phrases. p-adic analysis, Berkovich spaces, Ahlfors theory, covering surfaces.
The author gratefully acknowledges the support of a Miner D. Crary Research Fellowship from Amherst College.

must be the closure of the set of repelling periodic points. (The usual well known proofs of repelling density for rational functions do not extend to entire functions.)

In this paper, we will consider non-archimedean fields. Recall that a non-archimedean field is a field K equipped with a non-trivial absolute value $|\cdot|$ satisfying the ultrametric triangle inequality $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Standard examples of such fields include the p-adic rationals $\mathbb{Q}_p$ and various function fields. However, $\mathbb{Q}_p$ is not algebraically closed, and so we set the following notation.

K	a complete, algebraically closed non-archimedean field with absolute value $|\cdot|$
$\mathcal{O}_K$	the ring of integers $\{x \in K : |x| \leq 1\}$ of K
k	the residue field of K

For example, K could be $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$. Recall that the residue field k is defined to be $\mathcal{O}_K/\mathcal{M}_K$, where $\mathcal{M}_K$ is the maximal ideal $\{x \in K : |x| < 1\}$ of $\mathcal{O}_K$. We refer the reader to [17, 26] for treatises on non-archimedean analysis.

There have been numerous studies in recent decades of non-archimedean versions of Nevanlinna theory. In 1971, Adams and Straus [1] proved some non-archimedean Nevanlinna-style results using methods much simpler than a full Nevanlinna theory. More recently, a number of authors have developed a broader non-archimedean Nevanlinna theory, including analogues of the First and Second Main Theorems; see [13] or [21] for expositions, and [11, 12, 14, 15, 22, 27] for some of the original papers.

At the same time, there has also been a growing interest in the dynamics of non-archimedean rational and entire functions. Broad surveys can be found in [4, 5, 24, 25]. Although many of the fundamental results of complex dynamics have analogues in the non-archimedean setting, the question of non-archimedean repelling density remains open, even in the case of rational functions. There have been some partial results: Hsia [20] has shown that the Julia set of a rational function is contained in the closue of *all* periodic points, and Bézivin [9] has shown that repelling density follows if there is at least one repelling periodic fixed point. However, as discussed in the introduction to [6], there are serious obstacles to extending either result to prove repelling density completely.

Bearing Baker's complex result on repelling density in mind, as well as following the lead of the non-archimedean Nevanlinna theorists, the author presented a non-archimedean version of Ahlfors' Islands Theorem for holomorphic functions in [6]. In that case, only two islands, rather than three, were required. However, an extra hypothesis was also needed, essentially stating that the analogue of $L(f, r)$ is at some point larger than a constant which depends on the two islands. In this paper, we continue those investigations by presenting an analogue of the Islands Theorem for meromorphic functions in Theorem 5.2. We envisage that these non-archimedean islands theorems should be part of a non-archimedean theory of Ahlfors' covering surfaces which is yet to be developed.

The aforementioned Theorem 5.2 requires the theory of Berkovich spaces, including the Berkovich projective line $\mathcal{P}^1(K)$, which we shall discuss in Section 3. We give an abbreviated statement of the result here.

Main Theorem. (Non-archimedean Meromorphic Four Islands Theorem)
Let $U_1, U_2, U_3, U_4 \subseteq K \cup \{\infty\}$ be four disjoint open disks. Let ν_1 be a Berkovich point such that no connected component of $\mathcal{P}^1(K) \setminus \{\nu_1\}$ intersects more than two of U_1, U_2, U_3, U_4. Then there are real constants C_1, C_2 depending only on K and U_1, U_2, U_3, U_4 with following property.

Let f be a meromorphic function on $\{z \in K : |z| < 1\}$ such that $f^{\#}(0) > C_1$ and, for any point $\nu \in \mathcal{D}(0,1)$ in the open Berkovich disk such that $f_*(\nu) = \nu_1$, we have $L(f,\nu) \geq C_2$.

Then there is an open disk $U \subseteq D(0,1)$ such that f is one-to-one on U and $f(U) = U_i$ for some $i = 1, 2, 3, 4$.

Here, $f^{\#}$ is a non-archimedean version of the spherical derivative (see Definition 2.3 and equation (3.3)), and $L(f,\nu)$ is an analogue of the relative boundary length (see equation (4.1)). The other specialized notation, such as $\mathcal{D}(0,1)$, $\mathcal{P}^1(K)$, and $f_*(\nu)$, will be defined in Section 3. The full statement of the result, Theorem 5.2, includes precise descriptions of the constants C_1 and C_2; the sharpness of the statement and the constants will be considered in Examples 6.1 and 6.2.

In Section 1, we will recall some basic facts about the non-archimedean projective line $\mathbb{P}^1(K)$. In Section 2 we will review some standard results about meromorphic functions on non-archimedean disks. Section 3 is a summary of the fundamentals of the Berkovich theory. It is meant to be a self-contained introduction to the subject, for readers not familiar with it; proofs of the basic facts will be omitted. Section 3 also includes a number of Lemmas which will be needed for our main results. In Section 4, we will study some particular functions from Berkovich spaces to $\mathbb{R}$, including the quantity $L(f,\nu)$. Section 5 is devoted to the statement and proof of the main theorem and a corollary. Finally, we will present some examples and address the sharpness of Theorem 5.2 in Section 6.

The author would like to thank William Cherry for fruitful discussions on non-archimedean Nevanlinna and Ahlfors theory, and Jonathan Lubin for some clarification concerning Example 6.1.

1. The non-archimedean projective line

Let $\mathbb{P}^1(K)$ denote the projective line over K, with points represented in homogeneous coordinates by $[x,y]$, for $(x,y) \in K \times K \setminus \{(0,0\}$. We will usually identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by taking $[x,y]$ to $z = x/y$, with $[1,0]$ corresponding to $z = \infty$.

The metric on K induces a standard spherical metric on $\mathbb{P}^1(K)$, given by

$$\Delta(P_1, P_2) = \frac{|x_1 y_2 - x_2 y_1|}{\max\{|x_1|, |y_1|\} \max\{|x_2|, |y_2|\}},$$

where $P_i = [x_i, y_i]$. Clearly $0 \leq \Delta(P_1, P_2) \leq 1$. In affine coordinates,

$$\Delta(z_1, z_2) = \frac{|z_1 - z_2|}{\max\{1, |z_1|\} \max\{1, |z_2|\}}.$$

Note that for $z_1, z_2 \in \mathcal{O}$, we have $\Delta(z_1, z_2) = |z_1 - z_2|$. The topology on K induced by Δ is exactly the same as that induced by $|\cdot|$.

The group $\mathrm{PGL}(2,K)$ acts by linear fractional transformations on $\mathbb{P}^1(K)$. As on the Riemann sphere, given any six points $P_1,P_2,P_3,Q_1,Q_2,Q_3 \in \mathbb{P}^1(K)$, there is a unique $\eta \in \mathrm{PGL}(2,K)$ such that $\eta(P_i)=Q_i$ for all $i=1,2,3$. Of course, this map η need not preserve distances.

On the other hand, the subgroup $\mathrm{PGL}(2,\mathcal{O})$ of transformations $z \mapsto (az+b)/(cz+d)$ with $a,b,c,d \in \mathcal{O}$ and $|ad-bc|=1$ is distance-preserving with respect to Δ. (See [6], Section 1, for example.) That is, given $\eta \in \mathrm{PGL}(2,\mathcal{O})$ and $P_1,P_2 \in \mathbb{P}^1(K)$, we have

$$\Delta(\eta(P_1),\eta(P_2)) = \Delta(P_1,P_2).$$

It is easy to check that given any two points $P_1,P_2 \in \mathbb{P}^1(K)$, there is a distance-preserving map $\eta \in \mathrm{PGL}(2,\mathcal{O})$ such that $\eta(P_1)=P_2$; in fact, there are many such maps.

2. Holomorphic and meromorphic functions

For $a \in K$ and $r>0$, we will denote by $D(a,r)$ and $\overline{D}(a,r)$ the open disk and closed disk (respectively) of radius r about a. If $r \in |K^\times|$, then $D(a,r) \subsetneq \overline{D}(a,r)$, whereas the two sets coincide if $r \notin |K^\times|$. It is well known that all disks in K are both open and closed as topological sets, but we keep the labels "open disk" and "closed disk" because the two can behave differently under the action of holomorphic and meromorphic functions.

By ultrametricity, any point of a disk is a center; but because K is algebraically closed, the radius is well defined. That is, $D(a,r)=D(b,s)$ if and only if $r=s$ and $b \in D(a,r)$; the analogous statement also holds for closed disks.

Definition 2.1. Let $U \subseteq K$ be a disk.

a. Let $a \in U$ and let $g: U \to K$. We say g is *holomorphic* on U if we can write g as a power series

$$g(z) = \sum_{i=0}^{\infty} c_i (z-a)^i \in K[[z-a]]$$

which converges for all $z \in U$.

b. Let $f: U \to \mathbb{P}^1(K)$. We say f is *meromorphic on U* if f is continuous on U, and if we can write f in homogeneous coordinates as

$$f(z) = [g(z), h(z)]$$

for all z in some dense subset of U, where g and h are holomorphic on U.

Thus, a holomorphic function is not just locally analytic but rigid analytic, in that its defining power series converges on the whole disk. (All the rigid analysis in this paper will be hidden from view inside the results of [6] and [8], but we refer the interested reader to [10, 18] for detailed background on the subject.) As noted in [6], Section 2, holomorphicity is well defined, in the sense that if $a,b \in U$, and if g can be written as a convergent power series centered at a, then g can also be written as a convergent power series centered at b. Naturally, any holomorphic function f is also meromorphic, by choosing $g=f$ and $h=1$. Conversely, any meromorphic function which never takes on the value ∞ is in fact holomorphic.

Intuitively, a meromorphic function is simply the quotient of two holomorphic functions, as in complex analysis. The technical "dense subset" condition in Definition 2.1 is

required only because the holomorphic functions g and h may have common zeros. As observed in [23], it may not be possible to choose g and h to remove all common zeros if U is an open disk. Fortunately, this technicality will not affect us, because we will not be concerned with any specific representation g/h of a given meromorphic function f.

Derivatives of holomorphic and meromorphic functions are defined in the same way as in real and complex analysis, and they satisfy all the usual algebraic rules. In particular, if $f(z)$ is holomorphic or meromorphic on a disk U, then so is $f'(z)$.

If f is holomorphic on a disk U, and if $U' \subsetneq U$ is a smaller disk, then $f(U')$ is a disk. Moreover, $f(U')$ is open (respectively, closed) if and only if U' is open (respectively, closed); see, for example, [6], Lemma 2.2. The following lemma relates the radius of $f(U')$ to that of $f(U)$.

Lemma 2.2. *Let $a \in K$ and $r > 0$. Let f be a holomorphic function on the open disk $D(a, r)$ Then*

$$D(f(a), r \cdot |f'(a)|) \subseteq f(D(a, r))$$

with equality if f is one-to-one. The analogous result also holds for the closed disk $\overline{D}(a, r)$.

Proof. This follows immediately from [6], Lemma 2.2. □

As the holomorphic image of a disk is a disk, we are motivated to define a disk $V \subseteq \mathbb{P}^1(K)$ to be either a disk in K in the usual sense or the complement of a disk in K. (Thus, a disk containing ∞ is precisely the complement of a disk in K.) Equivalently, V is a disk in $\mathbb{P}^1(K)$ if and only if it is the image under some $\eta \in \mathrm{PGL}(2, K)$ of a disk in K. We say V is open (respectively, closed) if it is either an open (respectively, closed) disk in K or the complement of a closed (respectively, open) disk in K.

Given those definitions, if f is meromorphic on a disk $U \subseteq K$, and if $U' \subsetneq U$ is a strictly smaller disk, then $f(U')$ is either all of $\mathbb{P}^1(K)$ or else a disk V in $\mathbb{P}^1(K)$. Moreover, V is open (respectively, closed) if and only if U' is.

Note that a disk in $\mathbb{P}^1(K)$ is not the same as a set of the form

$$\{P \in \mathbb{P}^1(K) : \Delta(P, a) \leq r\} \quad \text{or} \quad \{P \in \mathbb{P}^1(K) : \Delta(P, a) < r\}. \tag{2.1}$$

Indeed, for any $r > 1$, the region $D(0, r)$ is a disk in $\mathbb{P}^1(K)$, but it cannot be written in the form of (2.1). The same is true of $\mathbb{P}^1(K) \setminus D(0, r)$ for any $r < 1$.

Even though the spherical metric is not appropriate for defining radii of disks, it can be made useful for defining derivatives, as follows.

Definition 2.3. Let $U \subset K$ be a disk, and let $f : U \to \mathbb{P}^1(K)$ be a meromorphic function. Let $a \in U$. The *spherical derivative of f at a* is

$$f^{\#}(a) = \begin{cases} \dfrac{|f'(a)|}{\max\{1, |f(a)|^2\}} & \text{if } f(a) \neq \infty, \\[2ex] \left(\dfrac{1}{f}\right)^{\#}(a) & \text{if } f(a) = \infty. \end{cases}$$

Note that $f^{\#}$ takes values in $[0, \infty) \subseteq \mathbb{R}$, not in K. The reader may verify that

$$f^{\#}(a) = \lim_{z \to a} \frac{\Delta(f(z), f(a))}{|z - a|}.$$

Furthermore, if $\eta \in \mathrm{PGL}(2,\mathcal{O})$, then because η preserves Δ, it is immediate that

$$(\eta \circ f)^{\#}(a) = f^{\#}(a).$$

We refer the reader to [17, 26] for more on holomorphic and meromorphic functions. For an abbreviated survey including a number of results relevant to this paper, see [6], Section 2.

3. The Berkovich disk and projective line

Our discussion of meromorphic functions on $D(0,1)$ will involve their action on larger spaces defined by Berkovich. We refer the reader to his papers, especially [8], for background on general Berkovich spaces and for proofs of most of their basic properties. For our purposes, the reader may find the exposition in [28] more useful, as it is specific to the case of disks and the projective line, which are all we need here. The same space for the projective line was independently discovered later by Rivera-Letelier [24, 25]; the set we will call $\mathcal{P}^1(K)$ is called $\mathbb{H} \cup \mathbb{P}^1(K)$ in his notation.

For $a \in K$ and $r \in |K^\times|$, the Berkovich disk $\overline{\mathcal{D}}(a,r)$ associated to the closed disk $\overline{D}(a,r)$ is defined as follows. Let $\mathcal{A}(a,r)$ be the ring of all holomorphic functions on $\overline{D}(a,r)$, with Gauss norm $\nu(a,r)$ given by

$$\|f\|_{\nu(a,r)} = \max\{|c_i|r^i : i \geq 0\}, \tag{3.1}$$

where $f(z) = \sum_{i=0}^{\infty} c_i(z-a)^i$. Intuitively, $\|f\|_{\nu(a,r)}$ is the generic value of $|f(x)|$ on $\overline{D}(a,r)$, in the sense that most $x \in \overline{D}(a,r)$ (i.e., all but those in finitely many open subdisks $D(b,r)$) satisfy $|f(x)| = \|f\|_{\nu(a,r)}$.

A *bounded multiplicative seminorm* ν on $\mathcal{A}(a,r)$ is a function $\|\cdot\|_\nu : \mathcal{A}(a,r) \to [0,\infty)$ such that for all $f, g \in \mathcal{A}(a,r)$,

i. $\|0\|_\nu = 0$,
ii. $\|1\|_\nu = 1$,
iii. $\|fg\|_\nu = \|f\|_\nu \cdot \|g\|_\nu$,
iv. $\|f+g\|_\nu \leq \max\{\|f\|_\nu, \|g\|_\nu\}$, and
v. $\|f\|_\nu \leq \|f\|_{\nu(a,r)}$.

(The above versions of properties (iv) and (v) are stronger than the usual definitions, but they are equivalent for our ring $\mathcal{A}(a,r)$, as shown in the first few pages of [28].) The function ν is called a seminorm because $\|f\|_\nu = 0$ need not necessarily imply that $f = 0$.

The Berkovich disk $\overline{\mathcal{D}}(a,r)$ is then defined to be the set of all bounded multiplicative seminorms on $\mathcal{A}(a,r)$, with the Gel'fond topology, which is the weakest topology such that

$$\{\nu \in \overline{\mathcal{D}}(a,r) : \|f\|_\nu < R\} \quad \text{and} \quad \{\nu \in \overline{\mathcal{D}}(a,r) : \|f\|_\nu > R\}$$

are open, for any $f \in \mathcal{A}(a,r)$ and any $R \in \mathbb{R}$. The reader may check that for fixed $f, g \in \mathcal{A}(a,r)$ and $R \in \mathbb{R}$, the set

$$\{\nu \in \overline{\mathcal{D}}(a,r) : \|f\|_\nu < R\|g\|_\nu\}$$

is also open. The space $\overline{\mathcal{D}}(a,r)$ is compact, Hausdorff, and path-connected [8]. Berkovich also showed that the points of $\overline{\mathcal{D}}(a,r)$ come in four types, which we now list.

There is a natural inclusion of $\overline{D}(a,r)$ in $\overline{\mathcal{D}}(a,r)$, as follows. If $x \in \overline{D}(a,r) \subseteq K$, then the value of the seminorm $\|f\|_x$ is defined to be simply $|f(x)|$. The corresponding points of $\overline{\mathcal{D}}(a,r)$ are called the *type I points*.

Meanwhile, for every $b \in \overline{D}(a,r)$ and every $s \in (0,r]$, the seminorm $\nu(b,s)$, which in this case is actually a norm, is defined exactly as in equation (3.1), but for the disk $\overline{D}(b,s)$. That is,

$$\|f\|_{\nu(b,s)} = \max\{|d_i|s^i : i \geq 0\}, \quad \text{where} \quad f(z) = \sum_{i=0}^{\infty} d_i(z-b)^i.$$

Note that this definition makes sense for any $s \in (0,r]$, not just $s \in |K^\times|$. Also note that the norm depends only on the disk $\overline{D}(b,s)$, not on the choice b of center. The corresponding point $\nu(b,s)$ of $\overline{\mathcal{D}}(a,r)$ is said to be of *type II* if $s \in |K^\times|$ and of *type III* otherwise.

From the type II and III points, one can begin to see how $\overline{\mathcal{D}}(a,r)$ is path-connected, at least between type I points. Given $b,c \in \overline{D}(a,r)$, the path from b to c starts at b, which we consider as a disk of radius zero. We increase the radius through a path of type II and III points of the form $\nu(b,s)$ until we get to $s = |b-c|$. Then $\nu(b,s) = \nu(c,s)$, and so we may decrease the radius s towards the new center c until we arrive at c itself.

Finally, there is one more class of points. A *type IV* point ν corresponds to a nested infinite sequence of disks $U_1 \supset U_2 \supset \cdots$ with empty intersection. (Such sequences may exist because K need not be maximally complete. The infimum of the radii of the U_i for such a sequence is always strictly positive.) The norm ν is simply the limit of the norms $\nu(U_i)$. (Of course, it is the limit norm ν, not the sequence $\{\nu(U_i)\}$, which is the type IV point; there are infinitely many equivalent sequences $\{U_i\}$ that approach any given type IV point.) The type IV points are needed to make $\overline{\mathcal{D}}(a,r)$ compact, but they will not be important in our discussions.

Intuitively, the space $\overline{\mathcal{D}}(a,r)$ looks like a tree branching out from the root point $\nu(a,r)$ with infinitely many branches at every type II point (which are dense in the tree), and with limbs ending at the type I and type IV points. The infinitely many branches at a type II point $\nu(b,s)$ correspond to the infinitely many open subdisks $D(c,s)$ of $\overline{D}(b,s)$ of the same radius, as well as (if $s < r$) one more branch corresponding to increasing the radius (i.e., corresponding to the disk at ∞). The type III points, meanwhile, are interior points with no branching.

If f is meromorphic on $\overline{D}(a,r)$ and $\nu \in \overline{\mathcal{D}}(a,r)$, then we may define $\|f\|_\nu$ to be $\|g\|_\nu/\|h\|_\nu$, where $f = g/h$ for $g,h \in \mathcal{A}(a,r)$. (Note that $\|f\|_\nu = \infty$ if and only if $\nu = b$ is type I and f has a pole at b.) As before, it is appropriate to think of $\|f\|_{\nu(a,r)}$ as the generic value of $|f(x)|$ for $x \in \overline{D}(a,r)$. The extended function $\|\cdot\|_\nu$ still satisfies properties (i)–(iv) of multiplicative seminorms, but we may no longer have $\|f\|_\nu \leq \|f\|_{\nu(a,r)}$.

For an open disk $D(a,r)$, we can also associate a Berkovich space $\mathcal{D}(a,r)$ by taking the union (really, the direct limit) of sets $\overline{\mathcal{D}}(a,r_i)$, where $r_i \nearrow r$. The resulting space is still path-connected, Hausdorff, and locally compact, but it is no longer compact. Although $\mathcal{D}(0,1)$ will be one of our main objects of study, we will understand it by considering the subspaces $\overline{\mathcal{D}}(a,r)$ described above, for $a \in D(0,1)$ and $0 < r < 1$.

We may also define the Berkovich projective line $\mathcal{P}^1(K)$ by glueing two copies of $\mathcal{D}(0,r)$ (for some $r > 1$) as follows. A type I point x on one copy with $1/r < |x| < r$ is identified with $1/x$ on the other copy. Meanwhile, a type II or III point $\nu(b,s)$ with $1/r < |b| < r$ is identified with $\nu(1/b, s/|b|^2)$, since $\overline{D}(1/b, s/|b|^2)$ is the image of $\overline{D}(b,s)$ under $z \mapsto 1/z$. A type IV point which is the limit of a sequence of type II points is mapped to the limit of the image of the sequence under $z \mapsto 1/z$.

Thus, $\mathcal{P}^1(K)$ looks like $\overline{\mathcal{D}}(0,1)$ with an extra copy of the open tree $\mathcal{D}(0,1)$ attached to the top (i.e., the ∞ end) of the point $\nu(0,1)$. The new top portion contains all points x of $\mathbb{P}^1(K)$ with $|x| > 1$, including ∞, as well as points $\nu(a,r)$ with $|a| > 1$ or $r > 1$. Like $\overline{\mathcal{D}}(0,1)$, the space $\mathcal{P}^1(K)$ is path-connected, Hausdorff, and compact.

Any disk in $\mathbb{P}^1(K)$ is associated with a unique point (of type II or III) of $\mathcal{P}^1(K)$. Indeed, any open or closed disk $\overline{D}(a,r)$ or $D(a,r)$ or its complement is associated with the point $\nu(a,r)$. Conversely, a type III point $\nu(a,r)$ is associated with exactly two disks, namely $D(a,r) = \overline{D}(a,r)$ and its complement. Meanwhile, a type II point $\nu(a,r)$ is associated with infinitely many disks: every open disk $D(b,r)$ for $b \in \overline{D}(a,r)$, the disk $\mathbb{P}^1(K) \setminus \overline{D}(a,r)$, and the complements of all these open disks. Although a type II or III point is associated with more than one disk, note that it is associated with exactly one closed disk which does not contain ∞. Thus, there is a one-to-one correspondence between type II and III points of $\mathcal{P}^1(K)$ and closed disks in K; the point $\nu(a,r)$ corresponds to $\overline{D}(a,r)$.

Borrowing from [25], we state the following definition.

Definition 3.1. Let X be a connected Berkovich space, let $W \subseteq X$ be a subset, and let $\nu_1 \in X$ be a point. We say that ν_1 *separates* W if W intersects more than one connected component of $X \setminus \{\nu_1\}$.

Following the intuition of the tree structure, it is not difficult to show for $X = \mathcal{P}^1(K)$, $X = \mathcal{D}(a,r)$, or $X = \overline{\mathcal{D}}(a,r)$, that if $\nu_1 \in X$ separates any set, then ν_1 must be type II or III.

We will usually consider separation in the case that the subset W contains only type I points. For example, $\nu(0,1)$ separates any subset of $\mathbb{P}^1(K)$ that contains both 0 and a point a with $|a| = 1$. However, $\nu(0,1)$ does not separate $D(0,1)$. Clearly, if $W \subseteq V$ and ν_1 separates W, then ν_1 also separates V.

As mentioned at the start of this section, a meromorphic function f on $\overline{D}(a,r)$ induces a function $f_* : \overline{\mathcal{D}}(a,r) \to \mathcal{P}^1(K)$. A fully rigorous derivation of f_* and its properties requires a description of general Berkovich spaces as locally ringed spaces with patches given by general Berkovich affinoids. We refer the reader to [8] or [28], Section B, for such a derivation; an equally rigorous derivation in a different style appears in [25], Section 4. We will now describe f_* precisely, but we will skip the proofs.

Using the more general Berkovich machinery, one can show that for each $\nu \in \mathbb{P}^1(K)$, there is a corresponding local ring $\mathcal{A}_\nu$ of functions f for which $\|f\|_\nu$ can be defined. If ν is of type II, III, or IV, then $\mathcal{A}_\nu$ contains $K(z)$, the ring of rational functions over K. If $\nu = b$ is of type I, then $\mathcal{A}_\nu$ contains all functions in $K(z)$ except those with poles at b; in that case, we may still talk about $\|f\|_b$ for functions f with a pole at b by defining $\|f\|_b = \infty$. The crucial fact from the Berkovich machinery is that ν is completely determined by its restriction to $K(z)$.

Thus, given f meromorphic on $\overline{D}(a,r)$ and $\nu \in \mathcal{D}(a,r)$, we define $f_*(\nu)$ to be the unique point (i.e., seminorm) in $\mathcal{P}^1(K)$ such that for all $h \in K(z)$,

$$\|h\|_{f_*(\nu)} = \|f \circ h\|_\nu. \tag{3.2}$$

The same definition applies to a meromorphic function f on $D(a,r)$. Similarly, if $\eta \in \mathrm{PGL}(2,K)$ and $\nu \in \mathcal{P}^1(K)$, we define $\eta_*(\nu)$ so that $\|h\|_{\eta_*(\nu)} = \|\eta \circ h\|_\nu$ for all $h \in K(z)$.

Equation (3.2) is somewhat unsatisfying at first; besides the fact that we have omitted the proof of existence and uniqueness of $f_*(\nu)$, the equation does not give much immediate insight into what f_* really looks like. Following [25], then, we present the following equivalent description.

Let $\overline{D}(a,r)$ be a closed disk with $r \in |K^\times|$. If f is holomorphic on $\overline{D}(a,r)$ and $\nu \in \overline{\mathcal{D}}(a,r)$ is a point of type II or III, then write $\nu = \nu(b,s)$ for some disk $D(b,s) \subsetneq \overline{D}(a,r)$. From Section 2 we know that $f(D(b,s))$ is an open disk; write $f(D(b,s)) = D(f(b),\sigma)$. Then f_* from equation (3.2) satisfies

$$f_*(\nu(b,s)) = \nu(f(b),\sigma).$$

More generally, given f meromorphic on $\overline{D}(a,r)$ and $\nu \in \overline{\mathcal{D}}(a,r)$ of type II or III, write $\nu = \nu(b,s)$ for some disk $D(b,s) \subsetneq D(a,r)$. It can be shown that there is a radius $s_0 < s$ such that for all s' with $s_0 < s' < s$, the image $f(D(b,s) \setminus \overline{D}(b,s'))$ of the annulus $D(b,s) \setminus \overline{D}(b,s')$ is itself an annulus of the form

$$D(\beta,\sigma) \setminus \overline{D}(\beta,\sigma') \quad \text{or} \quad D(\beta,\sigma') \setminus \overline{D}(\beta,\sigma),$$

where β and σ are fixed and do not vary with s'. Then it turns out that

$$f_*(\nu(b,s)) = \nu(\beta,\sigma).$$

It follows quickly from the definitions that f_* is a continuous function from one Berkovich space to another, and that f_* agrees with f at the type I points. Moreover, if f is a nonconstant meromorphic function, then f_* takes type I points to type I points, type II points to type II points, and so on.

Given f meromorphic on $\overline{D}(a,r)$, we can extend $f^\#$ from the type I points to all of $\overline{\mathcal{D}}(a,r)$ by setting

$$f^\#(\nu) = \frac{\|f'\|_\nu}{\max\{1, \|f\|_\nu^2\}}. \tag{3.3}$$

As was true for Definition 2.3, it is easy to show that if $\eta \in \mathrm{PGL}(2,\mathcal{O})$, then $(\eta \circ f)^\#(\nu) = f^\#(\nu)$.

Definition 3.2. Given a point $\nu(a,\rho) \in \mathcal{P}^1(K)$ of type II or III, define

$$r(\nu(a,\rho)) = \frac{\rho}{\max\{1, \rho^2, |a|^2\}} \tag{3.4}$$

to be the *spherical radius* of $\nu(a,\rho)$.

We leave it to the reader to verify that r is independent of the choice of a in $\overline{D}(a,\rho)$, and that r is a continuous function on the subset of $\mathcal{P}^1(K)$ on which it is defined. The spherical radius may also be defined at points of type I and IV by continuity, so that $r : \mathcal{P}^1(K) \to [0,\infty)$ is continuous. In fact, $r(\nu)$ is just $\exp[-\mathrm{dist}(\nu, \nu(0,1))]$, where

$\mathrm{dist}(\cdot,\cdot)$ is the metric on $\mathcal{P}^1(K) \setminus \mathbb{P}^1(K)$ which appears in [25], Section 3 as the metric on $\mathbb{H}$ and in [28], Section B as the "big model" metric.

If we restrict $r(\cdot)$ to $\mathcal{D}(0,1)$ or $\overline{\mathcal{D}}(0,1)$, then $r(\nu(a,\rho)) = \rho$ is the usual radius of the associated disk. Similarly, if $X = \mathcal{P}^1(K)$ and if $\nu = \nu(a,\rho)$ separates $\overline{D}(0,1)$ (which is to say that $\overline{D}(a,\rho) \subseteq \overline{D}(0,1)$), then $r(\nu)$ is again just the usual radius ρ. On the other hand, if $\nu(a,\rho)$ does not separate $\overline{D}(0,1)$, then $r(\nu) < \rho$. Formula (3.4) is chosen so that r is invariant under distance-preserving transformations, as the following lemma shows.

Lemma 3.3. *Let $\nu \in \mathcal{P}^1(K)$, and let $\eta \in \mathrm{PGL}(2,\mathcal{O})$. Then $r(\eta_*(\nu)) = r(\nu)$.*

Sketch of Proof. We may assume ν is not of type I or IV, as the result for types II and III will extend by continuity. Since $\mathrm{PGL}(2,\mathcal{O})$ is generated by maps of the form $z + A$ (for $|A| \leq 1$), Bz (for $|B| = 1$), and $1/z$, we may consider only such maps. The verification is trivial for $z + A$ and Bz. For $\eta(z) = 1/z$, we may write $\nu = \nu(a,\rho)$ with $\rho > 0$. If $0 \in \overline{D}(a,\rho)$, then $\nu = \nu(0,\rho)$ and $\eta_*(\nu) = \nu(0,1/\rho)$, from which the verification of the lemma is easy. On the other hand, if $0 \notin \overline{D}(a,\rho)$, then $|a| > \rho$, and $\eta_*(\nu) = \nu(1/a, \rho/|a|^2)$. The lemma then follows. $\square$

The remaining more specific lemmas will be needed to prove Theorem 5.2.

Lemma 3.4. *Let f be a nonconstant meromorphic function on $D(0,1)$, let $x \in D(0,1)$, let $0 < r < 1$, and set $\nu_1 = f_*(\nu(x,r)) \in \mathcal{P}^1(K)$. Then ν_1 separates $f(D(x, r+\varepsilon))$ for every $\varepsilon > 0$.*

Proof. Replacing f by $\eta \circ f$ for an appropriate $\eta \in \mathrm{PGL}(2,K)$, we may assume that $f(x) = 0$ and $\nu_1 = \nu(0,\rho)$ for some $\rho > 0$. If ν_1 separates $f(D(x,r))$, then we are done. If not, then $f(x) = 0$ forces $f(D(x,r)) \subseteq D(0,\rho)$; because $f_*(\nu(x,r)) = \nu_1$, we must have $f(D(x,r)) = D(0,\rho)$. For any given $\varepsilon > 0$, if f has a pole in $D(x, r+\varepsilon)$, then we are done; so we may assume that f is holomorphic on $D(x, r+\varepsilon)$. Since f is nonconstant with $f(x) = 0$, $\|f\|_{\nu(x,s)}$ must be a strictly increasing function of s for $0 < s < r + \varepsilon$; see equation (3.1). It follows that $\|f\|_{\nu(x,r+\varepsilon)} > \rho$, and therefore $f(D(x, r+\varepsilon))$ contains a point a with $|a| > \rho$, implying that ν_1 separates $f(D(x, r+\varepsilon))$. $\square$

Lemma 3.5. *Let f be meromorphic on $D(0,1)$, let $x \in D(0,1)$, let $0 < r < 1$, and let $\nu_1 \in \mathcal{P}^1(K)$. Then there are only finitely many points $\nu \in \mathcal{D}(x,r)$ such that $f_*(\nu) = \nu_1$.*

Proof. This Lemma follows easily from the machinery of [25], Section 4, but we include a direct proof for the convenience of the reader.

Write $f = g/h$ for g and h holomorphic. If $\nu_1 = a \in \mathbb{P}^1(K)$ is a type I point, then by a change of coordinates, we may assume $a = 0$. The lemma then holds for ν_1 by the finiteness of the Weierstrass degree of g on $D(x,r) \subsetneq D(0,1)$; see, for example, [6], Lemma 2.2.

If ν_1 is of type II or III, then let $a = f(x)$; there must be a point $b \in \mathbb{P}^1(K)$ such that ν_1 separates $\{a, b\}$. By a change of coordinates, we may assume $a = 0$ and $b = \infty$, so that $\nu_1 = \nu(0,R)$ for some $R > 0$. By the previous paragraph, there are only finitely many zeros and poles of f in $D(x,r)$.

If $f_*(\nu(y,s)) = \nu_1$ for some $\overline{D}(y,s) \subseteq D(x,r)$, then because $f(\overline{D}(y,s))$ is either $\mathbb{P}^1(K)$ or a closed disk associated with ν_1, there must be a root of $f = 0$ or $f = \infty$ in $\overline{D}(y,s)$. Thus, there can be only finitely many disjoint disks $\overline{D}(y_i,s_i)$ with $f_*(\nu(y_i,s_i)) = \nu_1$,

or else there would be infinitely many poles or zeros of f in $D(x,r)$, contradicting the previous paragraph.

Meanwhile, if $y \in D(x,r)$ and $0 < s_1 < s_2 < r$ with $f_*(\nu(y,s_i)) = \nu_1$ for $i = 1,2$, we claim that there must be a root y' of $f = 0$ or $f = \infty$ with $s_1 < |y' - y| \le s_2$. If not, then move y to 0 and write the holomorphic functions g, h as $g(z) = \sum_{j=0}^{\infty} a_j z^j$ and $h(z) = \sum_{j=0}^{\infty} b_j z^j$. Our assumption about the lack of roots implies that one term $a_m z^m$ of g and one term $b_n z^n$ of h is uniquely maximal in each sum for all $s_1 < |z| \le s_2$. Thus, for all such z,

$$|f(z) - cz^{\ell}| < |f(z)|,$$

where $c = a_m/b_n \in K$ and $\ell = m - n \in \mathbb{Z}$. Since $f_*(\nu(y,s_1)) = f_*(\nu(y,s_2)) = \nu(0,R)$, we must have $|cz^{\ell}| = R$ for all such z, which implies that $\ell = 0$ and $|c| = R$. In that case, however, $|f(z) - c| < R$ for all such z, meaning in particular that $|f(z) - c| < R$ for all $|z| = s_2$. Then $f_*(\nu(0,s_2)) \ne \nu(0,R)$, which is a contradiction and proves our claim.

Thus, any chain $\overline{D}(y_1,s_1) \supsetneq \overline{D}(y_2,s_2) \supsetneq \cdots$ of disks with $f_*(\nu(y_i,s_i)) = \nu_1$ must be finite, or else there would be infinitely many poles or zeros of f in $D(x,r)$. Together with the above fact that only finitely many disjoint disks $\overline{D}(y_i,s_i)$ can have $f_*(\nu(y_i,s_i)) = \nu_1$, it follows that there can be only finitely many $\nu \in \mathcal{D}(x,r)$ such that $f_*(\nu) = \nu_1$, as desired.

We will not need to consider the case that ν_1 is type IV in this paper, and we leave the proof of that case to the reader. $\square$

Lemma 3.6. *Let f be meromorphic on $D(x,r)$ for some $x \in K$ and $r > 0$, and let $\nu_1 \in \mathcal{P}^1(K)$. Then there is a small enough radius $r' > 0$ such that ν_1 does not separate $f(D(x,r'))$.*

Proof. By a change of coordinates, we may assume that $f(x) = 0$. The statement is vacuous if ν_1 is type I or IV, so we assume it is type II or III. We may therefore write $\nu_1 = \nu(a,R)$ for some $a \in K$, $R > 0$.

By Lemma 3.5, f has only finitely many poles in $D(x,r/2)$. We may therefore choose $0 < s < r/2$ such that there are no poles in $D(x,s)$, implying that f is holomorphic on $D(x,s)$. If $f(D(x,s)) \subseteq D(0,R)$, we are done. Otherwise, by [6], Lemma 2.6, there is a radius $r' \in (0,s]$ such that $f(D(x,r')) = D(0,R)$, which is not separated by ν_1. $\square$

Lemma 3.7. *Let f be meromorphic on $D(x,r)$ for some $x \in K$ and $r > 0$, and let $\nu_1 \in \mathcal{P}^1(K)$. Suppose that ν_1 separates $f(D(x,r))$. Then*

a. *there is some $\varepsilon > 0$ such that ν_1 separates $f(D(x,r-\varepsilon))$, and*
b. *there is a closed disk $\overline{D}(z,s) \subseteq D(x,r)$ such that $f_*(\nu(z,s)) = \nu_1$.*

Proof. Since ν_1 separates $f(D(x,r))$, there must be a point $y \in D(x,r)$ such that ν_1 separates $\{f(x), f(y)\}$. Let $\varepsilon = (r - |x-y|)/2 > 0$. Then $x, y \in D(x, r-\varepsilon)$, so that ν_1 separates $f(D(x,r-\varepsilon))$, proving part (a).

Recall that $\mathcal{D}(x,r)$ is connected and that $f_* : \mathcal{D}(x,r) \to \mathcal{P}^1(K)$ is continuous. Because $x, y \in \mathcal{D}(x,r)$ but $f(x)$ and $f(y)$ are in different connected components of $\mathcal{P}^1(K) \setminus \{\nu_1\}$, there must be some $\nu \in \mathcal{D}(x,r)$ such that $f_*(\nu) = \nu_1$. Note that ν_1, and therefore ν, must be points of type II or III, because ν_1 separates $\mathbb{P}^1(K)$. Thus, we may write $\nu = \nu(z,s)$ for some closed disk $\overline{D}(z,s) \subseteq D(x,r)$. $\square$

4. Functions on Berkovich spaces

Theorem 5.2 and its proof will rely heavily on certain functions from $\mathcal{D}(0,1)$ to $\mathbb{R}$. We have already seen the spherical radius function $r(\nu)$ defined in equation (3.4). In addition, if f is a holomorphic function on $D(0,1)$, then f induces another real-valued map, given by $\nu \mapsto \|f\|_\nu$. Such maps are continuous, because if $f = g/h$ with g, h holomorphic, then

$$\left\{\nu : \left\|\frac{g}{h}\right\|_\nu < R\right\} = \{\nu : \|g\|_\nu < R\|h\|_\nu\}$$

is open in the Gel'fond topology. The following lemma gives a more precise description of the behavior of $\|f\|_\nu$.

Lemma 4.1. *Let f be a nonconstant meromorphic function on $D(0,1)$, and let $a \in D(0,1)$. For $0 < r < 1$, define $F(r) = \|f\|_{\nu(a,r)}$. Then*

a. *$F : (0,1) \to (0,\infty)$ is a continuous function which is piecewise of the form $F(r) = cr^n$, for $c \in (0,\infty)$ and $n \in \mathbb{Z}$.*
b. *If $F(r) = cr^n$ on the interval $[r_1, r_2)$ for some $0 < r_1 < r_2 \leq 1$, then n is the number of zeros of f in $\overline{D}(a, r_1)$ less the number of poles, counting multiplicity of each.*
c. *For any fixed $0 < R < 1$, there are only finitely many radii $r \in (0, R]$ at which the value of the exponent n can change.*

Proof. This is a combination of Lemmas 4.4 and 4.5 of [6]. □

We now use the map $\nu \mapsto \|f\|_\nu$ to construct several other more specialized functions for use in our main theorem. Given a meromorphic function f on $D(0,1)$, define $L : \mathcal{D}(0,1) \to [0,\infty)$ by

$$L(\nu) = L(f,\nu) = r(\nu) \cdot f^{\#}(\nu) = \frac{r(\nu) \cdot \|f'\|_\nu}{\max\{1, \|f\|_\nu\}}. \tag{4.1}$$

We use the notation L because the above function is a non-archimedean analogue of Ahlfors' relative boundary length function $L(f,r)$. Note that for $\eta \in \mathrm{PGL}(2,\mathcal{O})$, we have $L(f,\nu) = L(\eta \circ f, \nu)$, because $f^{\#} = (\eta \circ f)^{\#}$.

Next, given f and a point $\alpha \in \overline{D}(0,1)$ with $\alpha \neq 0, 1$, define $G : \mathcal{D}(0,1) \to [0,\infty)$ by

$$G(\nu) = G(f,\alpha,\nu) = \frac{(r(\nu) \cdot \|f'\|_\nu)^2}{\|f\|_\nu \|f-\alpha\|_\nu \|f-1\|_\nu}. \tag{4.2}$$

The reason for the conditions on α will become clear in the proof of Theorem 5.2.

Equation (4.2) currently does not make sense if ν is one of the four points $0, \alpha, 1, \infty$ of type I. However, G extends naturally to all of $\mathcal{D}(0,1)$, as we now argue. By Lemma 4.1, for any fixed $a \in D(0,1)$, $G(\nu(a,r))$ is a continuous, piecewise monomial function of $0 < r < 1$, as is $L(\nu(a,r))$. In fact, by the same Lemma, for any $\|f\|_{\nu(a,r)}$, there is some $\varepsilon > 0$ such that the exponent n in $F(r) = cr^n$ is constant on $0 < r < \varepsilon$ and equal to the order of the zero (or negative the order of the pole) of f at a. It then follows fairly easily that the definition of G extends to all points of $\mathcal{D}(0,1)$, so that G is continuous and finite-valued on $\mathcal{D}(0,1)$.

Finally, given f and a point $b \in \mathbb{P}^1(K)$, then for any closed disk $\overline{D}(a,r) \subseteq D(0,1)$, define

$$\begin{aligned} N_b(a,r) &= \text{number of roots of } f = b \text{ in } \overline{D}(a,r), \text{ and} \\ N_{\text{ram}}(a,r) &= \text{number of ramification points of } f \text{ in } \overline{D}(a,r), \end{aligned} \tag{4.3}$$

where in each case "number" means the number of points, counted with multiplicity. Note that N_{ram} counts with multiplicity all points at which $f' = 0$; but it also counts ramification at all multiple poles. More precisely, if $f = g/h$ with g and h holomorphic, then N_{ram} counts (with multipicity) the zeros of $g'h - h'g$, less twice the number of common zeros of g and h. In addition, note that by Lemma 4.1, $G(\nu(a,r))$ is locally a monomial in r of degree

$$2 + 2N_{\text{ram}} - (N_0 + N_\alpha + N_1 + N_\infty),$$

because we may write

$$G(\nu) = \frac{(r(\nu) \cdot \|g'h - h'g\|_\nu)^2}{\|g\|_\nu \|g - \alpha h\|_\nu \|g - h\|_\nu \|h\|_\nu}.$$

We now list several properties of L, G, and N which will be useful in the proof of Theorem 5.2.

Lemma 4.2. *Let f be a meromorphic function on $D(0,1)$, let $\eta \in \mathrm{PGL}(2,K)$, and let $\nu \in \mathcal{D}(0,1)$ be a point of type II or III. Then*

$$\frac{L(f,\nu)}{r(f_*(\nu))} = \frac{L(\eta \circ f, \nu)}{r(\eta_*(f_*(\nu_1)))}.$$

Proof. Let $\nu_1 = f_*(\nu)$, and write $\nu_1 = \nu(a,\rho)$, with $a \in K$ and $\rho > 0$. Then $\|f\|_\nu = \max\{|a|, \rho\}$, so that

$$\frac{L(f,\nu)}{r(f_*(\nu))} = \frac{r(\nu)\|f'\|_\nu}{r(\nu_1)\max\{1, \|f\|_\nu^2\}} = \frac{r(\nu)\|f'\|_\nu \max\{1, \rho^2, |a|^2\}}{\rho \max\{1, \rho^2, |a|^2\}} = \frac{r(\nu)\|f'\|_\nu}{\rho}. \tag{4.4}$$

We may factor $\eta = \eta_1 \circ \eta_2$, where $\eta_1 \in \mathrm{PGL}(2, \mathcal{O})$ and $\eta_2(\infty) = \infty$. Since η_1 preserves r and L, we may assume without loss that $\eta = \eta_2$. Thus, $\eta(z) = B(z - a) + A$ for some $A, B \in K$ with $B \neq 0$.

We compute $\|(\eta \circ f)'\|_\nu = |B| \cdot \|f'\|_\nu$ (by the chain rule) and $\eta_*(\nu_1) = \nu(A, |B|\rho)$. Thus, by equation (4.4),

$$\frac{L(\eta \circ f, \nu)}{r(\eta_*(f_*(\nu)))} = \frac{r(\nu)\|(\eta \circ f)'\|_\nu}{|B|\rho} = \frac{r(\nu)\|f'\|_\nu}{\rho} = \frac{L(f,\nu)}{r(f_*(\nu))}.$$

□

The quantity $L(f,\nu)/r(f_*(\nu))$ in Lemma 4.2 is a measure of distortion; it generalizes the quantity δ which appeared in [6]. Intuitively, L itself measures the expected spherical radius of $f_*(\nu)$ based on the generic value of $f^\#$ at ν. Because $f^\#$ may be smaller than one would suspect (since z^p has small derivative pz^{p-1} in residue characteristic p), the actual spherical radius $r(f_*(\nu))$ of the image may be larger than L. Thus, the smaller the distortion ratio, the further the actual spherical radius is from that predicted by the derivative.

In this paper, we will only need to consider the quantity $L(f,\nu)/r(f_*(\nu))$ at points of type II or III. Nonetheless, we note here (and leave to the reader to verify) that the same quantity extends by continuity to all of $\mathcal{D}(0,1)$. If $\nu = x$ is of type I, then $L(f,\nu)/r(\nu_1)$ (which is 0/0 as written) turns out to be $|n|$, where $n \geq 1$ is the multiplicity with which x maps to $f(x)$.

The functions L and G are bounded above by 1, as the following lemma states.

Lemma 4.3. *Let f be a meromorphic function on $D(0,1)$, and let $\alpha \in \overline{D}(0,1) \setminus D(1,1)$, with $\alpha \neq 0$. Then for any $\nu \in \mathcal{D}(0,1)$,*

$$(L(f,\nu))^2 \leq G(f,\alpha,\nu) \leq 1.$$

Proof. By continuity, it suffices to show the result in the case that ν is type II. Recall from [6], Lemma 4.2, that $r(\nu)\|g'\|_\nu \leq \|g\|_\nu$ for a holomorphic function g on $\mathcal{D}(0,1)$. Thus, writing $f = g/h$ for g, h holomorphic on $D(0,1)$, we see that

$$r(\nu)\|f'\|_\nu = \frac{r(\nu)\|g'h - h'g\|_\nu}{\|h\|_\nu^2} \leq \frac{\|g\|_\nu}{\|h\|_\nu} \cdot \max\left\{\frac{r(\nu)\|g'\|_\nu}{\|g\|_\nu}, \frac{r(\nu)\|h'\|_\nu}{\|h\|_\nu}\right\} \leq \|f\|_\nu, \tag{4.5}$$

so that the same inequality holds for meromorphic functions as well.

Write $f_*(\nu) = \nu(a,R)$. If $f_*(\nu)$ separates $\mathbb{P}^1(K) \setminus \overline{D}(0,1)$, which is to say that $\max\{|a|, R\} > 1$, then

$$\|f\|_\nu = \|f-\alpha\|_\nu = \|f-1\|_\nu > 1,$$

so that by inequality (4.5),

$$G(\nu) = \frac{r(\nu)^2\|f'\|_\nu^2}{\|f\|_\nu^3} \leq \frac{1}{\|f\|_\nu} < 1.$$

The inequality $L^2 \leq G$ follows similarly.

If $f_*(\nu)$ separates $D(0,|\alpha|)$, which is to say that $\max\{|a|, R\} < |\alpha|$, then

$$\|f\|_\nu < \|f-\alpha\|_\nu = |\alpha|, \quad \text{and} \quad \|f-1\|_\nu = 1,$$

so that by inequality (4.5),

$$G(\nu) = \frac{r(\nu)^2\|f'\|_\nu^2}{|\alpha|\|f\|_\nu} < \left(\frac{r(\nu)\|f'\|_\nu}{\|f\|_\nu}\right)^2 \leq 1;$$

again, the $L^2 \leq G$ inequality also follows easily.

If $f_*(\nu)$ separates either $D(\alpha,|\alpha|)$ or $D(1,1)$, the verification is similar. Thus, the only remaining case is that α and R satisfy $|\alpha| \leq \max\{|a|, R\} \leq 1$, $|\alpha| \leq \max\{|a-\alpha|, R\} \leq 1$, and $\max\{|a-1|, R\} = 1$. In that case,

$$\|f\|_\nu = \|f-\alpha\|_\nu \leq 1 \quad \text{and} \quad \|f-1\|_\nu = 1,$$

so that

$$G(\nu) = \frac{r(\nu)^2\|f'\|_\nu^2}{\|f\|_\nu^2} \leq 1,$$

and $L(\nu)^2 = r(\nu)^2\|f'\|_\nu^2 \leq G(\nu)$. ☐

The following lemma, which uses the notation of equation (4.3), will be useful for choosing more useful centers for certain disks.

Lemma 4.4. *Let f be a meromorphic function on $D(0,1)$, and let $\alpha \in \overline{D}(0,1) \setminus \{0,1\}$. Fix $x \in D(0,1)$ and a radius $0 < R < 1$, and suppose that*

$$N_0(x,R) + N_\alpha(x,R) + N_1(x,R) + N_\infty(x,R) > 2N_{ram}(x,R).$$

Then there is a point $y \in \overline{D}(x,R)$ such that for every $r \in [0,R]$,

$$N_0(y,r) + N_\alpha(y,r) + N_1(y,r) + N_\infty(y,r) > 2N_{ram}(y,r).$$

Proof. Suppose not. Define

$$N_{\mathrm{tot}}(a,r) = N_0(a,r) + N_\alpha(a,r) + N_1(a,r) + N_\infty(a,r).$$

By Lemma 3.5, there are only finitely many roots $\{y_i\}_{i=1}^m$ of $f = 0, \alpha, 1, \infty$ in $\overline{D}(x,R)$. Then for each $i = 1, \dots, m$, there is some $r_i \in [0,R]$ such that $N_{\mathrm{tot}}(y_i, r_i) \le 2N_{\mathrm{ram}}(y_i, r_i)$. If any two of these disks intersect, then one contains the other, and so we may discard the smaller one. We are left with a finite set $\{\overline{D}(y_i', r_i')\}_{i=1}^{\ell}$ of pairwise disjoint disks in $\overline{D}(x,R)$, each of which satisfies $N_{\mathrm{tot}}(y_i', r_i') \le 2N_{\mathrm{ram}}(y_i', r_i')$, and which together contain all of the $\{y_i\}$. Thus,

$$N_{\mathrm{tot}}(x,R) = \sum_{i=1}^{\ell} N_{\mathrm{tot}}(y_i', r_i') \le 2 \sum_{i=1}^{\ell} N_{\mathrm{ram}}(y_i', r_i') \le 2N_{\mathrm{ram}}(x,R),$$

contradicting the hypotheses and hence proving the lemma. □

5. The Four Islands Theorem

We need the following specialized radius to define the key value μ which will appear in Theorem 5.2.

Definition 5.1. Let $U_1, U_2, U_3, U_4 \subseteq \mathbb{P}^1(K)$ be four disjoint open disks in $\mathbb{P}^1(K)$, and for each $i = 1,2,3,4$, fix a point $a_i \in U_i$. Choose a map $\eta_1 \in \mathrm{PGL}(2,K)$ such that $\eta_1(a_1) = 0$, $\eta_1(a_2) = \infty$, and $\eta_1(a_3) = 1$. Write $\eta_1(U_1) = D(0, r_1)$, and define

$$s_1 = \frac{r_1}{\min\{|\eta(a_2)|, |\eta(a_3)|, |\eta(a_4)|\}} = \frac{r_1}{\min\{1, |\eta(a_4)|\}}.$$

Define s_i similarly for $i = 2,3,4$ by mapping a_i to 0 and two of the other points to ∞ and 1. We define the *Ahlfors radius* of $\{U_1, U_2, U_3, U_4\}$ to be

$$s = \max\{s_1, s_2, s_3, s_4\}.$$

The reader may verify that the Ahlfors radius (and even the set $\{s_1, s_2, s_3, s_4\}$) is well defined, in the sense that it is independent of the ordering of the indices $1,2,3,4$ and the choice of the $\{a_i\}$. In addition, if $\tilde{\eta} \in \mathrm{PGL}(2,K)$, then the Ahlfors radius of $\{\tilde{\eta}(U_1), \tilde{\eta}(U_2), \tilde{\eta}(U_3), \tilde{\eta}(U_4)\}$ is the same as that of $\{U_1, U_2, U_3, U_4\}$. Note that $0 < s \le 1$.

Intuitively, each s_i is the ratio of the radius of U_i to the distance from U_i to the nearest other U_j. The Ahlfors radius is then the largest of these ratios. In the course of our proof, we will move the points $\{a_i\}$ to $\{0, \alpha, 1, \infty\}$ for some $\alpha \in \overline{D}(0,1) \setminus D(1,1)$ with $\alpha \neq 0$. To say that the Ahlfors radius of disks centered at those four points is at most s is to say that the four disks are contained in $D(0, |\alpha| s)$, $D(\alpha, |\alpha| s)$, $D(1, s)$, and $\mathbb{P}^1(K) \setminus \overline{D}(0, 1/s)$.

We are now prepared to state our main result.

Theorem 5.2. (Non-archimedean Meromorphic Four Islands Theorem)
Let K be a complete, algebraically closed non-archimedean field with residue field k, and let $p = \operatorname{char} k \geq 0$. If $p \geq 3$, define the real number

$$E_p = \sum_{i=1}^{\infty} \frac{1}{p^i - 1}.$$

Let $U_1, U_2, U_3, U_4 \subseteq \mathbb{P}^1(K)$ be four pairwise disjoint open disks. Let $\nu_1 \in \mathcal{P}^1(K)$ such that no connected component of $\mathcal{P}^1(K) \setminus \{\nu_1\}$ intersects more than two of U_1, U_2, U_3, U_4.
Let s be the Ahlfors radius of $\{U_1, U_2, U_3, U_4\}$. Set

$$\mu = \begin{cases} 0 & \text{if } \operatorname{char} k = 0, \\ s^{1/2} & \text{if } \operatorname{char} k = 2, \\ \min\left\{ s^{\left(\frac{1}{2} - \frac{1}{2p}\right)}, |p|^{-E_p} s^{1/2} \right\} & \text{if } \operatorname{char} k = p \geq 3, \end{cases}$$

and set

$$C_1 = \frac{1}{r(\nu_1)}, \quad \text{and} \quad C_2 = \mu \cdot r(\nu_1).$$

Let f be a meromorphic function in $D(0,1)$ such that

a. *$f^{\#}(0) > C_1$, and*
b. *for any point $\nu \in \mathcal{D}(0,1)$ such that $f_*(\nu) = \nu_1$, we have $L(f, \nu) \geq C_2$.*

Then there is an open disk $U \subseteq D(0,1)$ such that f is one-to-one on U and $f(U) = U_i$ for some $i = 1, 2, 3, 4$.

Remarks.

1. Because the Ahlfors radius satisfies $0 < s \leq 1$, it follows that $0 \leq \mu \leq 1$.
2. The statement of the theorem becomes stronger if μ is smaller, because more functions f will satisfy condition (b). Informally, then, the smaller μ is, the better.
3. Suppose $\operatorname{char} k = p \geq 3$. If $\operatorname{char} K = 0$, then μ decreases on the order of $s^{1/2}$ as s approaches 0. On the other hand, if $\operatorname{char} K = p$, then μ is $s^{(1/2 - 1/(2p))}$, which decreases more slowly.
4. In the $\operatorname{char} k = 0$ case, the choice of $\mu = 0$ above means that the value of C_2, and therefore the value of the Ahlfors radius s, is irrelevant; condition (b) becomes vacuous.
5. The lower bound of C_1 for $f^{\#}(0)$ in condition (a) is sharp, as we now observe. Choose $\lambda \in K$ with $|\lambda| \geq 1$. Let $f(z) = \lambda z$, let $\nu_1 = \nu(0, |\lambda|)$, and let each U_i be a disk of the form $D(a_i, \varepsilon)$, with $|a_i| = |\lambda|$ and $\varepsilon > 0$ as small as one wishes. Note that $f^{\#}(0) = C_1 = |\lambda|$, so that condition (a) just barely fails; however, all the other hypotheses of Theorem 5.2 hold. Nonetheless, the image $f(D(0,1))$ fails to intersect any U_i, let alone map a subdomain onto one of them.

   The sharpness of condition (b) is more subtle and will be considered in the examples of Section 6.

The following Lemma will appear in the final step of the proof of the Theorem. It is a slightly more complicated version of [6], Proposition 5.2.

Lemma 5.3. *Let K, k, p, and E_p be as in Theorem 5.2. Let $\alpha \in \overline{D}(0,1) \setminus D(1,1)$ with $\alpha \neq 0$, and let $\nu_1 \in \mathcal{P}^1(K)$. Suppose that no connected component of $\mathcal{P}^1(K) \setminus \{\nu_1\}$ contains more than two of $0, \alpha, 1, \infty$. Let f be a meromorphic function on $D(0,1)$, let $\mu \in [0,1]$, let $x \in D(0,1)$, and let $0 < R' < 1$. Suppose that*

(i) *ν_1 separates $f(D(x, R' + \varepsilon))$ for every $\varepsilon > 0$,*
(ii) *ν_1 does not separate $f(D(x, R'))$,*
(iii) *$N_0(x,r) + N_\alpha(x,r) + N_1(x,r) + N_\infty(x,r) > 2N_{ram}(x,r)$ for all $r \in [0, R']$, and*
(iv) *$G(f, \alpha, \nu(x, R')) \geq \mu^2$.*

Then there is an open disk $U \subseteq \overline{D}(x, R')$ such that f is one-to-one on U, and $f(U)$ is one of

$$D(0, |\alpha| s), \quad D(\alpha, |\alpha| s), \quad D(1, s), \quad \text{or} \quad \mathbb{P}^1(K) \setminus \overline{D}(0, 1/s),$$

where

$$s = \begin{cases} 1 & \text{if } p = 0 \\ \mu^2 & \text{if } p = 2 \\ \max\left\{|p|^{2E_p}\mu^2, \mu^{2p/(p-1)}\right\} & \text{if } p \geq 3. \end{cases}$$

Condition (i) is not actually required to prove the Lemma. It is stated here only for convenience, as all four conditions (i)–(iv) will figure prominently in the proof of Theorem 5.2. If ν_1 is a type II Berkovich point, then condition (i) is equivalent to the simpler statement that ν_1 separates $f(\overline{D}(0, r_0))$. However, the more complicated statement is required if ν_1 is type III.

Proof of Lemma 5.3. We may write $\nu_1 = \nu(0, \rho)$ for some $|\alpha| \leq \rho \leq 1$. By choosing r sufficiently small in property (iii), we have $f(x) \in \{0, \alpha, 1, \infty\}$.

We may assume that $f(x) = 0$. Indeed, if $f(x) = \alpha$, then let $\tilde{f} = \eta \circ f$, where $\eta(z) = (z - \alpha)/(1 - \alpha)$, which takes α to 0, fixes 1 and ∞, and takes 0 to $\tilde{\alpha} = \alpha/(\alpha - 1)$. The corresponding $\tilde{\nu}_1$ and $G(\tilde{f}, \tilde{\alpha}, \cdot)$ satisfy conditions (i)–(iv), and the disks $\eta(D(0, |\alpha| s))$, $\eta(D(\alpha, |\alpha| s)$, etc., are $D(\tilde{\alpha}, |\tilde{\alpha}|)$, $D(0, |\tilde{\alpha}|)$, etc., as appropriate. Similar arguments hold for $f(x) = \infty$ (with $\eta(z) = \alpha/z$) and for $f(x) = 1$ (with $\eta(z) = [\alpha(z-1)]/[(\alpha-1)z]$).

Since ν_1 does not separate $f(D(x, R'))$, and since $f(x) = 0$, f cannot have poles in $D(x, R')$. Similarly, it cannot take on the value 1 in $D(x, R')$. Thus, f and f' are holomorphic on $D(x, R')$, and $N_1(x,r) = N_\infty(x,r) = 0$ for all $0 < r < R'$.

Let $R'' = \inf\{0 < r \leq R' : \alpha \in f(\overline{D}(x, r))\}$ if this set is nonempty, or $R'' = R'$ if it is empty. For $R'' < r < R'$, we have $N_0(x,r) = N_\alpha(x,r)$, because any two points in the image of a holomorphic function have the same number of preimages; see [6], Lemma 2.2. For any such r, then, condition (iii) becomes

$$2N_{\mathrm{ram}}(x,r) + 2 - (N_0(x,r) + N_\alpha(x,r) + N_1(x,r) + N_\infty(x,r)) \leq 0, \tag{5.1}$$

because $N_0 + N_\alpha + N_1 + N_\infty = 2N_0$ is even. However, the left side of (5.1) is the local monomial degree of $G(\nu(x,r))$ at r. Thus, $G(\nu(x,r))$ is a locally constant or decreasing function of r on (R'', R'). By continuity and condition (iv), we have

$$G(\nu(x, R'')) \geq G(\nu(x, R')) \geq \mu^2.$$

Because $f(D(x,R'')) \subseteq D(0,|\alpha|)$, we have $\|f-\alpha\|_{\nu(x,r)} = |\alpha|$ and $\|f-1\|_{\nu(x,r)} = 1$ for all $0 < r \leq R''$. For all such r, then,

$$G(\nu(x,r)) = \frac{r^2\|f'\|^2_{\nu(x,r)}}{|\alpha|\cdot\|f\|_{\nu(x,r)}}, \tag{5.2}$$

and for $r < R''$, condition (iii) is

$$N_0(x,r) \geq 1 + 2N_{\mathrm{ram}}(x,r).$$

The remainder of the proof is identical with that of [6], Proposition 5.2, from the statement of Lemma 5.5 onward (pages 613–615). If we replace f by f/α, then expression (5.2) becomes $r^2\|f'\|^2_{\nu(x,r)}/\|f\|_{\nu(x,r)}$, which is exactly $F(r)$ in the notation of [6]. The proof in [6] ultimately concludes that this new f maps some disk $U = D(x,\tilde{R})$ one-to-one onto $D(0,s)$. Thus, the original map f maps U one-to-one onto $D(0,|\alpha|s)$, and we are done. ☐

Proof of Theorem 5.2. For each $i = 1,2,3,4$, choose a point $a_i \in U_i$. By hypothesis, we may assume that neither a_1 nor a_2 lies in the same component of $\mathcal{P}^1(K) \setminus \{\nu_1\}$ as either a_3 or a_4. Similarly, by exchanging the roles of $\{a_1,a_2\}$ and $\{a_3,a_4\}$ if necessary, we may assume that $f(0)$ does not lie in the same component as either a_3 or a_4.

Replacing f by $\eta_0 \circ f$ for some $\eta_0 \in \mathrm{PGL}(2,\mathcal{O})$, we may assume that $a_4 = \infty$; this change does not affect any of the hypotheses, by Section 1, by Lemma 4.2, and by the discussion following Definition 5.1. Note that ν_1 separates a_1 from both a_3 and $a_4 = \infty$, so that $\nu_1 = \nu(a_1,\rho)$ for some $0 < \rho \leq |a_3 - a_1|$. Also note that $a_2, f(0) \in \overline{D}(a_1,\rho) \setminus D(a_3,|a_3-a_1|)$. By the hypothesis that $f^{\#}(0) > 1/r(\nu_1) \geq \rho$, we compute

$$|f'(0)| = f^{\#}(0)\cdot\max\{1,|f(0)|^2\} > \rho.$$

Define $\eta \in \mathrm{PGL}(2,K)$ by $\eta(z) = (z-a_1)/(a_3-a_1)$, which is chosen so that $\eta(a_1) = 0$, $\eta(a_3) = 1$, and $\eta(a_4) = \infty$. Write $\tilde{f} = \eta\circ f$, $\tilde{\rho} = \rho/|a_3-a_1| \leq 1$, $\tilde{\nu}_1 = \eta_*(\nu_1)$, and $\alpha = \eta(a_2)$. Then $\tilde{\nu}_1 = \nu(0,\tilde{\rho})$, $\alpha \in \overline{D}(0,\tilde{\rho}) \setminus D(1,1)$, and $\tilde{f}(0) \in \overline{D}(0,\tilde{\rho})$. In addition,

$$|\tilde{f}'(0)| = \frac{|f'(0)|}{|a_3-a_1|} > \frac{\rho}{|a_3-a_1|} = \tilde{\rho}.$$

By Lemma 4.2, the hypothesis that $L(f,\nu) \geq \mu r(\nu_1)$ for any ν with $f_*(\nu) = \nu_1$ becomes $L(\tilde{f},\nu) \geq \mu r(\tilde{\nu}_1) = \mu\tilde{\rho}$ for any ν with $\tilde{f}_*(\nu) = \tilde{\nu}_1$. Any such ν satisfies

$$\|\tilde{f}\|_\nu = \|\tilde{f}-\alpha\|_\nu = \tilde{\rho}, \quad \text{and} \quad \|\tilde{f}-1\|_\nu = 1,$$

so that $L(\tilde{f},\nu) = r(\nu)\|\tilde{f}'\|_\nu$, and therefore

$$G(\tilde{f},\nu) = \frac{r(\nu)^2\|\tilde{f}'\|^2_\nu}{\|\tilde{f}\|_\nu\|\tilde{f}-\alpha\|_\nu\|\tilde{f}-1\|_\nu} = \frac{L(\tilde{f},\nu)^2}{\tilde{\rho}^2} \geq \mu^2.$$

Let

$$\mathcal{R}_0 = \Big\{0 < r < 1 : \nu_1 \text{ does not separate } \tilde{f}(D(0,r))\Big\},$$

which is nonempty, by Lemma 3.6. Let $r_0 = \sup\mathcal{R}_0 > 0$. If $r_0 = 1$, then $\tilde{f}(D(0,1)) \subseteq \overline{D}(0,\tilde{\rho})$. In that case, $\tilde{f}$ is holomorphic on $D(0,1)$, and by Lemma 2.2, $|\tilde{f}'(0)| \leq \tilde{\rho}$, which is a contradiction. Thus, $0 < r_0 < 1$.

If ν_1 separates $\tilde{f}(D(0,r_0))$, then by Lemma 3.7.a, ν_1 also separates $\tilde{f}(D(0,r_0-\varepsilon))$ for some $\varepsilon > 0$. In that case, $\mathcal{R}_0 \cap [r_0-\varepsilon, r_0] = \emptyset$, which contradicts the fact that r_0 is the supremum of $\mathcal{R}_0$. Thus, ν_1 does not separate $\tilde{f}(D(0,r_0))$, but for every $\varepsilon > 0$, ν_1 separates $\tilde{f}(D(0,r_0+\varepsilon))$. Moreover, $\tilde{f}(D(0,r_0)) \subseteq D(f(0),\tilde{\rho}) \subseteq \overline{D}(0,\tilde{\rho}) \setminus D(1,1)$, and $\tilde{f}$ is holomorphic on $(D(0,r_0))$.

From now on, we will no longer need the hypothesis that $f^{\#}(0) > C_1$. Writing f in place of $\tilde{f}$, then, we have $0 < r_0 < 1$, $0 < \rho \leq 1$, $\nu_1 = \nu(0,\rho)$, $\alpha \in \overline{D}(0,\rho) \setminus D(1,1)$ with $\alpha \neq 0$, and a meromorphic function f on $D(0,1)$ such that

- f is holomorphic on $D(0,r_0)$,
- $f(D(0,r_0)) \subseteq \overline{D}(0,\rho) \setminus D(1,1)$,
- $|f'(0)| > \rho$,
- ν_1 does not separate $f(D(0,r_0))$,
- ν_1 separates $f(D(0,r_0+\varepsilon))$ for all $\varepsilon > 0$, and
- for every $\nu \in \mathcal{D}(0,1)$ such that $f_*(\nu) = \nu_1$, we have $G(\nu) \geq \mu^2$.

We wish to show that f maps some open disk $U \subseteq D(0,1)$ one-to-one onto one of $D(0,s/|\alpha|)$, $D(\alpha,s/|\alpha|)$, $D(1,s)$, or $\mathbb{P}^1(K) \setminus \overline{D}(0,1/s)$.

Since $f(D(0,r_0)) \subseteq \overline{D}(0,\rho) \setminus D(1,1)$, we have

$$\|f\|_{\nu(0,r_0)}, \|f-\alpha\|_{\nu(0,r_0)} \leq \rho, \quad \text{and} \quad \|f-1\|_\nu = 1.$$

Moreover, since f (and hence f') is holomorphic on $D(0,r_0)$, we have $\|f'\|_{\nu(0,r_0)} \geq |f'(0)|$; see equation (3.1). Thus,

$$G(\nu(0,r_0)) = \frac{r_0^2\|f'\|^2_{\nu(0,r_0)}}{\|f\|_{\nu(0,r_0)}\|f-\alpha\|_{\nu(0,r_0)}\|f-1\|_{\nu(0,r_0)}} \geq \frac{|f'(0)|^2}{\rho^2} \cdot r_0^2. \tag{5.3}$$

Let

$$\mathcal{R} = \{r \in [r_0,1) : N_0(0,r) + N_\alpha(0,r) + N_1(0,r) + N_\infty(0,r) > 2N_{\mathrm{ram}}(0,r)\}.$$

As noted in Section 4, $G(\nu(0,r))$ is locally a monomial function of r of degree

$$2 + 2N_{\mathrm{ram}}(0,r) - [N_0(0,r) + N_\alpha(0,r) + N_1(0,r) + N_\infty(0,r)].$$

Therefore, a radius $r \in [r_0,1)$ is in $\mathcal{R}$ if and only if $G(\nu(0,\cdot))$ is of degree strictly less than two at r.

We claim that $\mathcal{R} \neq \emptyset$. Indeed, if $\mathcal{R} = \emptyset$, then by Lemma 4.1, $G(\nu(0,\cdot))$ is a continuous function which is piecewise monomial, and always of degree at least two, on $[r_0,1)$. By inequality (5.3), it follows that

$$G(\nu(0,r)) \geq \left(\frac{|f'(0)|}{\rho}\right)^2 r^2$$

for all $r \in [r_0,1)$. Since $|f'(0)| > \rho$, there must be some such r for which $G(\nu(0,r)) > 1$, contradicting Lemma 4.3 and proving the claim.

Let $R \in \mathcal{R}$. Note that $R \geq r_0$; thus, ν_1 separates $f(\overline{D}(0,R+\varepsilon))$ for every $\varepsilon > 0$. Moreover, by Lemma 4.4, we may choose $y \in \overline{D}(0,R)$ such that for every $r \in [0,R]$,

$$N_0(y,r) + N_\alpha(y,r) + N_1(y,r) + N_\infty(y,r) > 2N_{\mathrm{ram}}(y,r).$$

We wish to find a (possibly different) point $x \in \overline{D}(0,R)$ and a (possibly smaller) radius R' so that the pair (x,R') satisfies properties (i)–(iv) of Lemma 5.3.

We will do so by an inductive process. We begin with $y_1 = y$ and an auxiliary radius $R_1 = R$. We have just observed that properties (i) and (iii) already apply to the pair (y_1, R_1). At each step $n \geq 1$, given a point y_n and an auxiliary radius R_n, we will define the radius R'_n. We will prove that all of the desired properties hold for (y_n, R'_n) except possibly condition (iv). If that condition fails, we will construct a new point y_{n+1} and a new auxiliary radius R_{n+1}, and the process will repeat. We will then prove that there must eventually be some $n \geq 1$ for which condition (iv) holds.

The process is as follows. At step $n \geq 1$, we are given y_n and an auxiliary radius R_n such that properties (i) and (iii) apply to the pair (y_n, R_n). Define

$$\mathcal{R}'_n = \{r \in (0, R_n] : \nu_1 \text{ separates } f(D(y_n, r + \varepsilon)) \text{ for every } \varepsilon > 0\},$$

which is nonempty because $R_n \in \mathcal{R}'_n$. Let $R'_n = \inf \mathcal{R}'_n$. Observe that $R'_n > 0$, by Lemma 3.6. By definition of R'_n and the properties of (y_n, R_n), properties (i) and (iii) apply to the pair (y_n, R'_n). In addition, property (ii) applies to the pair, by Lemma 3.7.a. If $G(\nu(y_n, R'_n)) \geq \mu^2$, then property (iv) holds, and our process finishes by setting $(x, R') = (y_n, R'_n)$. We may therefore assume that $G(\nu(y_n, R'_n)) < \mu^2$.

We claim there is a disk $\overline{D}(z_{n+1}, R''_{n+1}) \subseteq \overline{D}(y_n, R'_n)$ such that $f_*(\nu(z_{n+1}, R''_{n+1})) = \nu_1$. By definition of R'_n, we know that ν_1 separates $f(D(y_n, r))$ for any $r \in (R'_n, 1]$. Hence, by Lemma 3.7.b, for any such r, there is a disk $\overline{D}(z, r'') \subseteq D(y_n, r)$ such that $f_*(\nu(z, r'')) = \nu_1$. On the other hand, by Lemma 3.5, there are only finitely many such disks $\overline{D}(z, r'')$ in $D(y_n, (R'_n + 1)/2) \subsetneq D(0, 1)$. If no such disks were contained in $\overline{D}(y_n, R'_n)$, then letting $r > R'_n$ be the minimum distance from such a disk to y_n, the conclusion of Lemma 3.7.b would fail for $D(y_n, r)$, contradicting the fact that ν_1 separates $f(D(y_n, r))$. Thus, our claim is proved, and the desired disk $\overline{D}(z_{n+1}, R''_{n+1})$ exists.

Because $f_*(\nu(z_{n+1}, R''_{n+1})) = \nu_1$, we have $G(\nu(z_{n+1}, R''_{n+1})) \geq \mu^2$. By our assumption that $G(\nu(y_n, R'_n)) < \mu^2$, we have $\overline{D}(z_{n+1}, R''_{n+1}) \subsetneq \overline{D}(y_n, R'_n)$; in particular, $R''_{n+1} < R'_n$.

Define $g(r) = G(\nu(z_{n+1}, r))$, so that g is continuous on the interval $[R''_{n+1}, R'_n]$, with $g(R''_{n+1}) \geq \mu^2 > g(R'_n)$. Therefore, g has an absolute maximum at some radius $R_{n+1} \in [R''_{n+1}, R'_n)$. Without loss, we may assume R_{n+1} is the largest radius in $[R''_{n+1}, R'_n]$ for which g attains its maximum. Thus, g is strictly decreasing on $[R_{n+1}, R_{n+1} + \varepsilon]$ for some $\varepsilon > 0$; by Lemma 4.1, we have

$$N_0(z_{n+1}, R_{n+1}) + N_\alpha(z_{n+1}, R_{n+1}) + N_1(z_{n+1}, R_{n+1}) + N_\infty(z_{n+1}, R_{n+1}) > 2N_{\mathrm{ram}}(z_{n+1}, R_{n+1}).$$

By Lemma 4.4, there is a point $y_{n+1} \in D(x, R)$ such that for every $r \in [0, R_{n+1}]$,

$$N_0(y_{n+1}, r) + N_\alpha(y_{n+1}, r) + N_1(y_{n+1}, r) + N_\infty(y_{n+1}, r) > 2N_{\mathrm{ram}}(y_{n+1}, r).$$

By Lemma 3.4, ν_1 separates $f(D(z_{n+1}, R''_{n+1} + \varepsilon))$ for every $\varepsilon > 0$. Since $\overline{D}(z_{n+1}, R''_{n+1}) \subseteq \overline{D}(y_{n+1}, R_{n+1})$, it follows that ν_1 separates $f(D(y_{n+1}, R_{n+1} + \varepsilon))$ for every $\varepsilon > 0$. Thus, the pair (y_{n+1}, R_{n+1}) satisfies properties (i) and (iii) above, so that our inductive process may repeat.

To show that the process must eventually end, we first claim that $y_n \notin \overline{D}(y_{n+1}, R'_{n+1})$ for every $n \geq 1$. Otherwise, because $y_{n+1} \in \overline{D}(y_n, R'_n)$ and $R'_{n+1} \leq R_{n+1} < R'_n$, we would have $D(y_n, R'_n) = D(y_{n+1}, R'_n)$. However, ν_1 separates $f(D(y_{n+1}, R'_n))$, by condition (i) for (y_{n+1}, R'_{n+1}). At the same time, ν_1 does not separate $f(D(y_n, R'_n))$, by condition (ii) for (y_n, R'_n). This contradiction proves the claim. In particular, all of the $\{y_n\}$ are distinct.

Observe that by choosing r sufficiently small in property (iii), each y_n must have $f(y_n) \in \{0, \alpha, 1, \infty\}$. By Lemma 3.5, there are only finitely many such points y_n in $\overline{D}(0, R)$; hence, the process must eventually stop. Thus, we obtain a pair (x, R') satisfying properties (i)–(iv) of Lemma 5.3. By the same Lemma, then, we are done. ☐

Corollary 5.4. *Let K, k, p, and E_p be as in Theorem 5.2. Let $U_1, U_2, U_3, U_4 \subseteq \mathbb{P}^1(K)$ be four pairwise disjoint open disks. Let $\nu_1 \in \mathcal{P}^1(K)$ such that no connected component of $\mathcal{P}^1(K) \setminus \{\nu_1\}$ intersects more than two of U_1, U_2, U_3, U_4. Define C_2 as in Theorem 5.2.*

Let f be a nonconstant meromorphic function on K such that for any point $\nu \in \mathcal{P}^1(K) \setminus \{\infty\}$ for which $f_*(\nu) = \nu_1$, we have $L(f, \nu) \geq C_2$. Then there is an open disk $U \subseteq D(0, 1)$ such that f is one-to-one on U and $f(U) = U_i$ for some $i = 1, 2, 3, 4$.

Proof. If $\operatorname{char} K = 0$, then the hypothesis that f is nonconstant implies that there is some $x \in K$ such that $f^\#(x) > 0$.

On the other hand, if $\operatorname{char} K = p > 0$, then there are many nonconstant functions for which $f'(z) = 0$; any function which can be written as $f(z) = g(z^p)$ with g meromorphic has this property. To avoid this situation, we first claim that ν_1 separates $f(D(0, r))$ for some $r > 0$.

If not, change coordinates so that $f(0) = 0$ and $\nu_1 = \nu(0, \rho)$ for some $\rho > 0$. Then f is holomorphic on K with image contained in $D(0, \rho)$. Because f is nonconstant, Lemma 4.1 implies that $\|f\|_{\nu(0,r)} \geq cr^n$ for some $c > 0$ and $n \geq 1$. Thus, we may choose $r > 0$ large enough that $\|f\|_{\nu(0,r)} > \rho$, which is a contradiction and proves our claim.

By Lemma 3.7.b, there is some $\nu \in \mathcal{D}(0, r)$ such that $f_*(\nu) = \nu_1$. Since $\operatorname{char} K > 0$, we have $C_2 > 0$; by hypothesis, then, $L(f, \nu) > 0$, so that $f^\#(\nu) > 0$. Therefore there is a point $x \in K$ such that $f^\#(x) > 0$.

In any characteristic, then, given the point x above, define C_1 as in Theorem 5.2. By an affine change of coordinates (moving x to 0 and scaling appropriately), we may assume that $f^\#(0) > C_1$. The result then follows by restricting f to the open unit disk and invoking Theorem 5.2. ☐

6. Examples

Theorem 5.2 differs from its complex counterpart in several noticeable ways. First and foremost, only four islands are required, as opposed to the five in the complex case. As observed in [6], Example 6, it would be impossible to reduce the number further, to three islands. On the other hand, in the case of positive residue characteristic, the non-archimedean theorem requires the extra condition that $L(f, \nu) \geq C_2$ for any ν mapping to ν_1. A similar condition, that $L(f, \nu(0, r)) \geq C_2$ for *some* r, is required for the holomorphic version [6]. In this section, we present examples to illustrate both that the lower bound of C_2 is essentially sharp, and that it is not enough to assume only one ν mapping to ν_1 satisfies the inequality, even if the number of islands is increased.

For the sharpness of the constant C_2, in Example 6.1 we will consider only the case that $\operatorname{char} k = p > 0 = \operatorname{char} K$; the map in question is very much analogous to that of [6], Example 5. As in [6], we show that the bounds given in Theorem 5.2 are sharp if $p = 2$ and almost sharp (except possibly for the constant $|p|^{-E_p}$) if $p \geq 3$. If $\operatorname{char} k = 0$, then the lower bound C_2 is vacuously sharp. We conjecture that examples analogous to [6], Examples 3 and 4, would prove the sharpness of C_2 in the cases that $\operatorname{char} K = p > 0$.

Example 6.1. Suppose $\operatorname{char} K = 0$ by $\operatorname{char} k = p \geq 2$. (For example, suppose $K = \mathbb{C}_p$, the completion of an algebraic closure of the p-adic rationals $\mathbb{Q}_p$.) Let E be an elliptic curve defined over K with identity point O, and let $n \geq 1$ be an integer. Assume that E has good ordinary reduction. (The conclusions we will reach also hold for multiplicative reduction, but that case is slightly more complicated because there are a number of different points ν which will map to the point ν_1 we will choose shortly.) Let E_1 be the of points which map to $\overline{O}$ under reduction.

If we identify E_1 with the open unit disk, then by the characteristics of K and K, and because the corresponding formal group has height 1, there are p-torsion points $\{P_1, \ldots, P_{p-1}\}$ at distance $|p|^{1/(p-1)}$ from O in E_1. Moreover, there are no nontrivial torsion points closer than $|p|^{(p-1)}$ to O.

The multiplication-by-n map $[n] : E \to E$ has the property that $[n](-P) = -[n](P)$ for any point P on E. Meanwhile, the group $\{\pm 1\}$ acts on the curve E (with -1 taking P to $-P$) with quotient $\mathbb{P}^1$. It follows that there is a map $f_n : \mathbb{P}^1 \to \mathbb{P}^1$ for which

$$\begin{tikzcd} E \arrow[r, "{[n]}"] \arrow[d, "h"'] & E \arrow[d, "h"'] \\ \mathbb{P}^1 \arrow[r, "f_n"] & \mathbb{P}^1 \end{tikzcd}$$

commutes, where h is quotient map. The function f_n is known as a Lattès map to dynamicists. It is a rational function of (geometric) degree n^2.

Let a_1, a_2, a_3, a_4 be the images under h of the 2-torsion points $E[2]$ of E. Note that $E[2]$ is the set of ramification points of h. For convenience, choose coordinates on $\mathbb{P}^1$ so that $h(O) = a_1 = 0$ and so that $h(E_1) = D(0,1)$. Let $\nu_1 = \nu(0,1)$; then because E has good reduction, we have $(f_n)_*(\nu) = \nu_1$ if and only if $\nu = \nu_1$. Furthermore, an examination of the map $[n]$ restricted to the formal group E_1 shows that $L(f_n, \nu_1) = |n|$. Let $\mu = |n|$.

Let $C_n = E[2n] \setminus E[2]$ be the set of $2n$-torsion points which are not 2-torsion, and let $B_n \subseteq \mathbb{P}^1(K)$ be the image of C_n under h. By considering the ramification of the map h, and knowing that f_n must have exactly $2 \deg f_n - 2 = 2n^2 - 2$ critical points, it is easy to check that B_n is the set of critical points of f_n, and that each point of B_n maps 2-to-1 to its image. One can also check that $f_n'(0) = n^2$.

Because h maps E_1 two-to-one onto $D(0,1)$, it is not difficult to show that $|h(P_i)| = |p|^{2/(p-1)}$. Similarly, the lack of nontrivial torsion points in E_1 at distance less than $|p|^{1/(p-1)}$ from O implies that

$$(\{a_2, a_3, a_4\} \cup B_n) \cap D(0, |p|^{2/(p-1)}) = \emptyset. \tag{6.1}$$

In fact, if $p \nmid n$ and $p \geq 3$, then $(\{a_2, a_3, a_4\} \cup B_n) \cap D(0,1) = \emptyset$.

Meanwhile, $B_n \cup \{a_1, a_2, a_3, a_4\}$ is precisely the preimage of $\{a_1, a_2, a_3, a_4\}$ under f_n. Thus, if we choose U_1, U_2, U_3, U_4 to be disjoint disks containing a_1, a_2, a_3, a_4 in Theorem 5.2, then the only disks U that could map one-to-one onto any U_i would have to contain exactly one of a_1, a_2, a_3, a_4 and cannot intersect B_n. By translating on E by 2-torsion points, it suffices to consider only one-to-one mappings of f_n from a disk U containing $a_1 = 0$ to $U_1 \subseteq D(0,1)$. In fact, the preimage disk U we should consider is the largest disk about 0 which contains no points in $B_n \cup \{a_2, a_3, a_4\}$. If $p|n$ or $p = 2$, then

by equation (6.1), this disk is $U = D(0, |p|^{2/(p-1)})$. By Lemma 2.2, the largest possible one-to-one image disk is $U_1 = D(0, |n|^2 \cdot |p|^{2/(p-1)})$, since $|f_n'(0)| = |n|^2$. On the other hand, if $p \nmid n$ and $p \geq 3$, the domain disk is $U = D(0,1)$, and its image is $U_1 = D(0,1)$.

If $p = 2$, then there is one nontrivial 2-torsion point P_1 in E_1. Its image $h(P_1) = a_2$ is the point α of Section 5. In addition, no component of $\mathbb{P}^1(K) \setminus \{\nu_1\}$ contains more than two of a_1, a_2, a_3, a_4; thus, f_n satisfies the conditions of Corollary 5.4. Since $|\alpha| = |p|^{2/(p-1)}$, the radius of the image disk U_1 described in the previous paragraph is exactly $|\alpha| \cdot \mu^2$. Thus, the Ahlfors radius is $s = \mu^2$, which is exactly the lower bound in Lemma 5.3.

If $p \geq 3$, we have $|\alpha| = 1$, since none of $\{a_2, a_3, a_4\}$ lie in $D(0,1)$. If $p \nmid n$, then $\mu = 1$, so that the lower bound from Lemma 5.3 for the radius s of the image disk is 1, which is exactly the radius of the disk U_1 found above. Finally, if $p|n$, then the image disk has radius $s = \mu^2 |p|^{2/(p-1)}$, which is only slightly larger than the Lemma 5.3 lower bound of $\mu^2 |p|^{2E_p}$.

Our final example will illustrate that having only one ν for which $f_*(\nu) = \nu_1$ with $L(f,\nu)$ bounded below by some fixed amount is not enough to guarantee an islands theorem, regardless of how many islands there are, how small they are, or how small the lower bound on $L(f,\nu)$ is. As in Example 6.1, Example 6.2 is only for the case that $\operatorname{char} k > \operatorname{char} K = 0$. Of course, if $\operatorname{char} k = 0$, then the condition that $L(f,\nu) \geq 0$ is vacuous, as previously noted, so there will be no counterexamples, as we already know the theorem is already true without hypothesis (b) in that case. On the other hand, if $\operatorname{char} K = p > 0$, then we imagine that examples similar to the following one may be constructed.

Example 6.2. Assume that $\operatorname{char} k = p > 0 = \operatorname{char} K$. For any integer $N \geq 0$ and any radius $0 < s \leq 1$, we select $N+2$ islands as follows. Set $a_0 = 0 \in K$. For each $i = 1, \ldots, N$, choose $a_i \in K$ with $|a_i| = 1$ and, for each $i \neq j$, $|a_i - a_j| = 1$. The first $N+1$ islands will be the open disks $D(a_i, s)$, with $i = 0, \ldots, N$. The final island will be $\mathbb{P}^1(K) \setminus \overline{D}(0, 1/s)$, which is the open disk of spherical radius s centered at ∞. Let $\nu_1 = \nu(0,1)$, which separates each island from every other.

Pick $n \geq 1$ large enough so that $|p^n| < s$, and choose $b \in K$ so that $0 < |b| < s$. Let $c = -b^{1+p^n}$.

For each $i = 1, \ldots, N$, define

$$f_i(z) = \frac{(z-a_i)^{p^n} + c}{(z-a_i)^{p^n}} = 1 + \frac{c}{(z-a_i)^{p^n}}.$$

Then, define

$$f_0(z) = \frac{z^{1+p^n} + c}{z^{p^n}} = z + \frac{c}{z^{p^n}}, \quad \text{and} \quad f(z) = \prod_{i=0}^{N} f_i(z).$$

The reader may check that $f_*(\nu_1) = \nu_1$ and $L(f, \nu(0,1)) = 1$; recall from Lemma 4.3 that this is the maximum value L could ever attain.

The only preimages of ∞ are $a_0, \ldots, a_N$, all of which are critical points. Thus, the island at ∞ has no one-to-one preimages. In addition, if $|z - a_i| \geq 1$ for all $i = 0, \ldots, N$,

then it is easy to see that $|f(z)-a_i| \geq 1$ also. In particular, any preimages of the islands $D(a_i, s)$ must lie in the disks $D(a_j, 1)$.

Next, observe that there are $1+p^n$ preimages of 0 in $D(0,1)$, namely the roots of $f_0(z)=0$, which are all of the form $\zeta^j b$, where ζ is a primitive $(1+p^n)$-root of unity. In particular, the largest open disk about any such root which contains no other such roots has radius $|b|$. It is easy to compute that $|f'(\zeta^j b)| = 1$, and therefore the image of that largest open disk is a disk of radius $|b| < s$. Thus, the island at 0 is not a one-to-one image of a disk inside $D(0,1)$.

Similarly, for any fixed $i = 1, \ldots, N$, there are p^n preimages of 0 in $D(a_i, 1)$, namely the roots of $(z-a_i)^{p^n} = -c$. Those roots are of the form $x = a_i + \omega^j d$, where ω is a p^n-root of unity, and d is a p^n-root of $-c$. Any ω^j is distance $|p|^{1/(p-1)}$ from the nearest other ω^ℓ, so that the largest disk containing exactly one root of $(z-a_i)^{p^n} = -c$ has radius $|p|^{1/(p-1)}|c|^{1/p^n}$. We can compute that $|f'(x)| = |p|^n \cdot |c|^{-1/p^n}$, so that the largest possible one-to-one image disk has radius $|p|^{n+(1/(p-1))} < |p|^n < s$. Thus, there are no one-to-one preimages of the island at 0 anywhere in K.

For $1 \leq i \leq N$, the preimages of a_i in $D(0,1)$ must be points $x \in D(0,1)$ satisfying $|f_0(x)| = 1$. Since $|x| < 1$, this must mean $|x| = |c|^{1/p^n}$. Because of the pole at 0, the largest open disk about x which could conceivably map onto the island $D(a_i, s)$ must have radius at most $|c|^{1/p^n}$. (In fact, it will have slightly smaller radius than that, but the bound of $|c|^{1/p^n}$ will suffice for our purposes.) Some computation using the above value for $|x|$ shows that $|f'(x)| \leq \max\{1, |p|^n |c|^{-1/p^n}\}$, and therefore the largest possible one-to-one image disk has radius

$$\max\left\{|c|^{1/p^n}, |p|^n\right\} \leq \max\left\{|b|, |p|^n\right\} < s,$$

which fails to cover the island.

Before considering the final case, observe that

$$f(z) - z = z \cdot \left[1 - \left(1 + \frac{c}{z^{1+p^n}}\right) \prod_{j=1}^{N} \left(1 + \frac{c}{(z-a_j)^{p^n}}\right)\right],$$

so that

$$|f(z) - z| \leq |z| \cdot \max\left\{\left|\frac{c}{z^{1+p^n}}\right|, \left|\frac{c}{(z-a_1)^{p^n}}\right|, \ldots \left|\frac{c}{(z-a_N)^{p^n}}\right|\right\}.$$

It follows that if $|c|^{1/(1+p^n)} < |x - a_j| < 1$ for some $j = 1, \ldots, N$ and some $x \in K$, then $|f(x) - x| < |x - a_j|$. In particular, no such x can have image $f(x)$ in any of the $N+2$ islands.

We are now ready to consider the final possibility, that there is a preimage of an island $D(a_i, s)$ in the disk $D(a_j, 1)$ for some $i, j \in \{1, \ldots, N\}$, where i and j may or may not be equal. In such a case, we have a point x in $D(x, a_j)$ with $f(x) = a_i$. This means that $|f_j(x)| = 1$, so that $|x - a_j| \geq |c|^{1/p^n}$. From this bound, it follows that $|f'(x)| \leq \max\{1, |p^n|/|x - a_j|\}$. Meanwhile, the largest possible disk mapping onto $D(a_i, s)$ cannot contain the pole at a_j, so that its radius must be at most $|x - a_j|$. Thus, the largest possible one-to-one image disk about a_i has radius at most

$$|f'(x)| \cdot |x - a_j| \leq \max\{|x - a_j|, |p^n|\} \leq \max\{|c|^{1/(1+p^n)}, |p^n|\} = \max\{|b|, |p^n|\} < s,$$

where the second inequality is by the previous paragraph. Thus, none of the N islands has a one-to-one preimage anywhere in K, in spite of the fact that $f_*(\nu_1) = \nu_1$ with $L(f, \nu(0,1)) = 1$.

An examination of the proof of Theorem 5.2 applied to Example 6.2 reveals why the hypothesis that *every* (or at least many) ν mapping to ν_1 must have $L(f, \nu) \geq \mu$. For that choice of f, we start from $R = 1$ (or whatever smaller radius $\overline{D}(0, 1)$ is moved to after f is scaled as described in the proof of Corollary 5.4) and move inward, searching for a disk satisfying conditions (i)–(iv) of Lemma 5.3. Properties (i), (iii), and (iv) already apply to $\overline{D}(0, R)$, but property (ii) does not. Lemma 4.4 would select the new center y_1 to be one of the roots of $f = a_i$ in $D(a_i, 1)$ for some $i = 0, \ldots, N$; the inductive process would begin with $R_1 = 1$ and y_1 being one such root. The minimal radius R_1' would be the smallest radius about y_1 for which $f(\overline{D}(y_1, R_1'))$ contained points outside $D(a_i, 1)$. That is, $R_1' = |y_1 - a_i|$, which we saw to be at most $|b|$. Even though the inequality of property (iii) holds, the degree of r in G is positive (in fact, equal to 1) for $r \in [R_1', R_1]$, so that as r shrinks from R_1 down to R_1', G also shrinks from 1 down to $R_1' \leq |b|$. Thus, although f is at last one-to-one on $D(y_1, R_1')$, the image is too small because there are poles too close to y_1, and the value of G (as well as L, along with it) has shrunk too much. There are type II points ν which separate the smaller disk $D(y_1, R_1')$, but without hypothesis (b.) of the Theorem, their G-values are too small. Thus, if we try to shrink to a disk $D(z_2, R_2'')$ according to the algorithm, we have no guarantee that G increases, and therefore we have no guarantee that property (iii) holds.

On the other hand, a modified version of the same example, with $f_0(z) = z + c/z^{p^n}$ replaced by $f_0(z) = z + c/z^{p^n - 1}$, has the same pathology of points ν mapping to ν_1 with small $G(\nu)$ in each of the disks $D(a_i, 1)$ for $i = 1, \ldots, N$. This time, however, for $i = 0$, there is one extra ν mapping to ν_1 in $D(0, 1)$ which *does* satisfy $G(\nu) = 1$. As a result, the conclusion of Theorem 5.2 holds for the modified example, because there are disks in $D(0, 1)$ which map one-to-one onto, say, $D(a_1, 1)$. Thus, it is conceivable that some condition weaker than "$L(f, \nu) \geq C_2$ for *all* ν mapping to ν_1" but stronger than "$L(f, \nu) \geq C_2$ for *some* ν mapping to ν_1" would suffice. We leave the existence of such a condition as an open question.

References

[1] W. Adams and E. Straus, Non-archimedean analytic functions taking the same values at the same points, *Illinois J. Math.* **15** (1971), 418–424.

[2] L. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* **65** (1935), 157–194.

[3] I.N. Baker, Repulsive fixpoints of entire functions, *Math. Z.* **104** (1968), 252–256.

[4] R. Benedetto, p-adic dynamics and Sullivan's No Wandering Domains Theorem, *Compositio Math.* **122** (2000), 281–298.

[5] R. Benedetto, Components and periodic points in non-archimedean dynamics, *Proc. London Math. Soc.* (3) **84** (2002), 231–256.

[6] R. Benedetto, Non-archimedean holomorphic maps and the Ahlfors Islands Theorem, *Amer. J. Math.*, **125** (2003), 581–622.

[7] W. Bergweiler, A new proof of the Ahlfors five islands theorem, *J. Anal. Math.* **76** (1998), 337–347.

[8] V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields,* Amer. Math. Soc., Providence, 1990.

[9] J.-P. Bézivin, Sur les points périodiques des applications rationnelles en analyse ultramétrique, *Acta Arith.* **100** (2001), 63–74.

[10] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry,* Springer-Verlag Berlin, 1984.
[11] A. Boutabaa, Théorie de Nevanlinna p-adique, *Manuscripta Math.* **67** (1990), 251–269.
[12] A. Boutabaa and A. Escassut, Nevanlinna theory in characteristic p and applications, *Analysis and applications—ISAAC 2001 (Berlin)*, Kluwer, Dordrecht, 2003, 97–107.
[13] W. Cherry, A survey of Nevanlinna theory over non-Archimedean fields, *Bull. Hong Kong Math. Soc.* **1** (1997), 235–249.
[14] W. Cherry and Z. Ye, Non-Archimedean Nevanlinna theory in several variables and the non-Archimedean Nevanlinna inverse problem, *Trans. Amer. Math. Soc.* **349** (1997), 5043–5071.
[15] C. Corrales-Rodrigáñez, Nevanlinna theory on the p-adic plane, *Ann. Polon. Math.* **57** (1992), 135–147.
[16] J. Dufresnoy, Sur les domaines couvertes par les valeurs d'une fonction méromorphe ou algébroïde, *Ann. Sci. École Norm. Sup. (3)* **58** (1941), 179–259.
[17] A. Escassut, *Analytic Elements in p-adic Analysis,* World Scientific, Singapore, 1995.
[18] D. Goss, A short introduction to rigid analytic spaces, *The arithmetic of function fields*, Columbus, OH, 1991, 131–141.
[19] W. Hayman, *Meromorphic Functions*, Oxford Univ. Press, London, 1964.
[20] L.-C. Hsia, Closure of periodic points over a nonarchimedean field, *J. London Math. Soc.* (2) **62** (2000), 685–700.
[21] P.-C. Hu and C.-C. Yang, *Meromorphic Functions over Non-Archimedean Fields*, Mathematics and its Applications 522, Kluwer, Dordrecht, 2000.
[22] Hà Huy Khoái and My Vinh Quang, On p-adic Nevanlinna theory, *Complex Analysis, Joensuu 1987*, *Lecture Notes in Math.* **1351**, Springer, Berlin, 1988, 146–158.
[23] M. Lazard, Les zéros des fonctions analytiques sur un corps valué complet, *Inst. Hautes Études Sci. Publ. Math.* **14** (1962) 47–75.
[24] J. Rivera-Letelier, Dynamique des fonctions rationnelles sur des corps locaux, *Astérisque* **287** (2003), 147–230.
[25] J. Rivera-Letelier, Espace hyperbolique p-adique et dynamique de fonctions rationnelles, *Compositio Math.* **138** (2003), 199–231.
[26] A. Robert, *A Course in p-adic Analysis*, Springer-Verlag, New York, 2000.
[27] M. Ru, A note on p-adic Nevanlinna theory, *Proc. Amer. Math. Soc.* **129** (2001), 1263–1269.
[28] R. Rumely, Analysis on the Berkovich line, preprint, 2004. Available online at `http://www.math.uga.edu/~ rr/`.
[29] L. Zalcman, A heuristic principle in complex function theory, *Amer. Math. Monthly* **82** (1975), 813–817.

Department of Mathematics and Computer Science, Amherst College, Amherst, MA 01002, USA
E-mail address: `rlb@cs.amherst.edu`
URL: `http://www.cs.amherst.edu/~rlb`